THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and twenty-eighth regular meeting of the Society was held in New York City on Saturday, April 28. The following fifty members attended the sessions:

Dr. Grace Andrews, Professor G. A. Bliss, Professor Maxime Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. W. H. Bussey, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Dr. W. S. Dennett, Professor L. P. Eisenhart, Professor B. F. Finkel, Professor T. S. Fiske, Professor G. H. Hallett, Professor E. R. Hedrick, Dr. L. I. Hewes, Mr. A. M. Hiltebeitel, Professor E. V. Huntington, Mr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Professor E. O. Lovett, Dr. Emory McClintock, Professor James Maclay, Professor H. P. Manning, Professor Max Mason, Mr. A. R. Maxson, Professor W. F. Osgood, Professor James Pierpont, Mr. R. G. D. Richardson, Mr. W. H. Roever, Miss Ida M. Schottenfels, Mr. C. H. Sisam, Dr. Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Dr. H. F. Stecker, Dr. C. E. Stromquist, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor H. S. White, Dr. Ruth G. Wood.

President W. F. Osgood occupied the chair. The Council announced the election of the following persons to membership in the Society: Rev. R. D. Carmichael, Hartselle, Ala.; Mr. F. L. Griffin, University of Chicago; Mr. W. R. Longley, University of Chicago; Mr. W. D. MacMillan, University of Chicago; Mr. F. W. Owens, Evanston Academy; Dr. J. J. Quinn, High School, Warren, Pa.; Mr. W. J. Risley, University of Illinois; Dr. R. P. Stephens, Wesleyan University; Mr. J. D. Suter, Iowa State College; Mr. A. M. Wilson, McKinley High School, St. Louis, Mo. Eighteen applications for membership in the Society were received.

Professor W. F. Osgood was elected a member of the Editorial Committee of the *Transactions*, to succeed Professor E. W. Brown, who retires after seven years' service covering the entire period of existence of that journal.

The by-laws were amended to provide that only members of at least four years' standing shall be permitted to compound life membership.

In the interval between the sessions the members lunched together, and the informal dinner in the evening, attended by some thirty members, afforded another welcome opportunity for conference and renewal of acquaintance.

The following papers were read at this meeting:

(1) Professor G. A. MILLER: "Groups in which all the operators are contained in a series of subgroups such that any two of them have only identity in common."

(2) Mr. W. H. ROEVER: "Lines of force illustrated by rotating carriage wheels."

(3) Mr. W. H. ROEVER: "Systems of lines of force whose differential equations take Bernoulli's form in polar coordinates."

(4) Professor VIRGIL SNYDER: "On twisted curves contained in a linear complex."

(5) Mr. G. E. WAHLIN: "On the number of classes of binary quadratic forms and the ideals of a quadratic body."

(6) Mr. R. G. D. RICHARDSON: "On the fundamental theorem in the reduction of multiple integrals."

(7) Professor JAMES PIERPONT: "On the area of curved surfaces."

(8) Professor E. R. HEDRICK: "Functions and their derivatives on given assemblages."

(9) Professor E. R. HEDRICK: "Lipschitz's condition in the case of impiicit functions."

(10) Professor MAX MASON: "A necessary condition for an extremum of a double integral."

(11) Professor G. A. BLISS: "An invariant of the calculus of variations corresponding to geodesic curvature."

(12) Dr. EDWARD KASNER: "A generalization of conformal representation."

(13) Dr. EDWARD KASNER: "Velocity curves in the dynamics of a particle."

(14) Professor J. W. YOUNG: "A generalization of a problem of Tchebychev."

(15) Professor C. J. KEYSER: "Concerning the bond uniting elements into a space."

(16) Dr. C. N. HASKINS: "Note on the differential invariants of a plane."

(17) Mr. E. C. COLPITTS: "On twisted quintic curves."

(18) Mr. W. C. BREUKE: "On the differentiation of trigonometric series" (preliminary communication).

(19) Dr. I. E. RABINOVITCH: "The necessary and sufficient kinematic axioms of geometry."

Mr. Wahlin was introduced by Professor Pierpont, Dr. Rabinovitch by Professor Cole; Mr. Colpitts's paper was communicated through Professer Snyder. In the absence of the authors, the papers of Professor Miller and Professor Keyser were read by title, those of Dr. Haskins and Mr. Colpitts were presented by Professor Snyder, and Mr. Breuke's paper by Professor Bôcher. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Miller's paper appears in full in the present number of the BULLETIN.

2. Mr. Roever illustrated, by means of the crossings of spokes of two wagon wheels rotating in parallel planes, the lines of force of a system composed of two sources (of the same or different intensities and of the same or different algebraic signs). The hubs of the wheels correspond to the sources, and the ratio of the angular velocities of the wheels is the negative reciprocal of the ratio of the intensities of the sources. In particular, he considered the case in which a rolling wheel is viewed through a picket fence, and explained why it is that the wheel appears to turn backwards.

3. In this paper Mr. Roever considered the field of force which results from the combination of the fields I and II described below.

Field I. At any point P the direction of the force due to this field passes through a fixed point O, and the magnitude of the force is represented by f_1.

Field II. At any point P the direction of the force due to this field is parallel to the fixed direction OY, and the magnitude is represented by f_2.

If we denote by r the distance OP, and by θ the complement of the angle YOP, the differential equation of the lines of force of the resulting field is

$$\frac{dr}{d\theta} - \tan\theta \cdot r = -\frac{f_1}{f_2} r \sec\theta. \tag{1}$$

If

$$\frac{f_1}{f_2} = r^n F(\theta),$$

equation (1) takes Bernoulli's form $(dy/dx + P(x)\cdot y = Q(x)\cdot y^n)$.

As applications, Mr. Roever solved the following problems.

(1) Let it be required to find the lines of force of the system which is composed of a uniform electric field of intensity 2κ and an electrified point of charge M. Here

$$f_1 = \frac{m}{r^2}, \quad f_2 = 2\kappa, \quad \frac{f_1}{f_2} = \frac{m}{2\kappa}\frac{1}{r^2}.$$

(2) Let it be required to find the lines of force of the system which is composed of a uniform electric field of intensity 2κ, and an electrified straight line (of infinite length) of charge λ per unit length, the straight line being perpendicular to the lines of force of the uniform field. Here

$$f_1 = \frac{2\lambda}{r}, \quad f_2 = 2\kappa, \quad \frac{f_1}{f_2} = \frac{\lambda}{\kappa}\frac{1}{r}.$$

(3) Assuming the earth to be a rigid sphere with its center of mass at its center of figure, let it be required to find the lines of force of the system which is composed of the gravitational field and the field due to the centrifugal force of rotation. This is the field of force which determines the direction of the plumb line. Here

$$f_1 = \frac{M}{r^2}, \quad f_2 = \omega^2 y = \omega^2 r \sin\theta, \quad \frac{f_1}{f_2} = \frac{M}{\omega^2}\frac{1}{r^3 \sin\theta},$$

where ω is the angular velocity of rotation and M is the mass of the earth.

4. Curves belonging to a linear complex can be depicted into developables containing a conic by means of the Noether point-line transformation. The latter are transformed by duality into curves lying on a quadric cone. In Professor Snyder's paper the details of these transformations are worked out, and a classification of curves contained in a linear complex is given. The simplest such curve of genus one is of order six, of genus two of order seven, of genus five of order eight, etc.

5. Mr. Wahlin presented a simultaneous development of the two theories in question, showing how, by applying the ideal theory to the forms, we are able to simplify the evaluation of the number of classes. He also gave an application of the ideals to the determination of the number of genera of forms.

6. In the *Journal de Mathématiques*, series 4, volume 8 (1892), de la Vallée-Poussin proposes the following problems: Is the existence of the double integral over a plane field T,

$$\int_T f(x, y)dxdy,$$

and of the iterated integral

$$\int_Y dy \int_X f(x, y)dx$$

a sufficient condition for the equality of these integrals? He proves that such is the case if certain restrictions are put on the arrangement of the infinities of the function $f(x, y)$. In the same journal, series 5, volume 5 (1899), he devotes a memoir to this problem, and shows that under certain limitations of the function the theorem is true. He states, however, that in his opinion the theorem is true without these restrictions. Mr. Richardson proved that the existence of these integrals is a sufficient condition for their equality and that the condition of uniform or regular convergence of the integral

$$\int_X f(x, y)dx$$

is entirely unnecessary. It is shown that this theorem is also true for the generalized integrals introduced by Professor Pierpont. Further, if the fields of integration are $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, of $m + n$, m, and n dimensions respectively, then the existence of

$$\int_{\mathfrak{A}} f(x_1, x_2, \cdots, x_{m+n}) \quad \text{and} \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f(x_1, x_2, \cdots, x_{m+n})$$

is a sufficient condition for their equality.

7. Professor Pierpont extended the results of Stolz (*Transactions*, volume 3 (1902), page 23) in a two-fold manner.

First, he considered surfaces whose coordinates do not have first partial derivatives, finite or infinite, at a discrete aggregate of points in the parametric field. Secondly, he considered a much wider system of inscribed polyhedra than is employed by Stolz.

8. In this paper Professor Hedrick discusses the properties of functions of one or more real variables which are defined for the values of the independent variable or variables which constitute a given assemblage. The continuity of such functions, the possibility of adjoining further definition, definitions and theorems concerning continuity, approach to limits, etc., form the first part of the paper.

The discussion of derivatives of such functions leads to an important notion of the derivative of any function on an assemblage, and in particular to the notion of sequence derivatives. After a discussion of these concepts, a general theorem is proved, namely, that if a continuous sequence derivative exists, the ordinary derivative exists and is continuous.

An extension of these results to a generalized notion of derivative and to the concept of a jacobian as an assemblage derivative in space closes the paper.

9. In this paper Professor Hedrick develops the analogon to Lipschitz's condition for the existence of solutions of systems of differential equations, and shows the form which the condition assumes in the case of the solutions of implicit equations. The form of proof given recently by Goursat is used as a basis, but the conditions are materially weakened. In particular the Goursat condition which requires the existence of $\partial F(x, y)/\partial y$ (for an implicit equation $F(x, y) = 0$ which is to be solved for y) is replaced by the weaker condition that the difference quotient

$$Q(x, \bar{y}, \bar{\bar{y}}) \equiv \frac{F(x, \bar{y}) - F(x, \bar{\bar{y}})}{\bar{y} - \bar{\bar{y}}}$$

should lie between two positive numbers (or two negative numbers) for all points near the known point (x_0, y_0) which is on $F(x, y) = 0$. The generalization to n equations is given in several forms.

10. If a function $z(x, y)$ gives an extreme value (maximum or minimum) to the integral

$$J = \int\int f(x, y, z, p, q)\, dxdy \qquad (p = z_x, q = z_y),$$

then the first variation of J must vanish, and the second variation must have the same sign for all allowed variations of z. Professor Mason showed that the second variation could be made positive or negative at pleasure, unless

$$f_{pp} f_{qq} - f_{pq}^2 \geqq 0.$$

This inequality is accordingly a necessary condition for an extremum.

11. The geodesic curvature of a curve on a given surface is usually defined by means of the projection of the given curve on a tangent plane to the surface, but it may also be defined in other ways. For example, if two geodesic lines are drawn tangent to the curve at neighboring points A and A', they will in general general intersect at a point B. The limit as A' approaches A of the ratio of the angle at B between the two geodesics, divided by the length of the arc AA', is the geodesic curvature at the point A.

In the paper of Professor Bliss a generalization of geodesic curvature is found by making use of the definition just given. A problem of the calculus of variations is considered in which the integral has the form

$$I = \int f(x, y, \tau) \sqrt{x'^2 + y'^2}\, dt,$$

where x and y are functions of t defining the given curve, and τ is the angle between the tangent to the curve and the x-axis. In a previous paper the writer has given a generalization corresponding to this integral. If two extremals are taken tangent to the given curve at neighboring points A and A', they will in general intersect, and the generalized angle between them may be determined. The limit of the ratio of this angle to the value of I taken along the arc AA', as A' approaches A, is the desired curvature. It turns out to be an invariant under point transformation.

12. In a previous paper, Dr. Kasner has shown that if a surface is mapped conformally on a plane, the geodesics through any point are pictured by curves whose circles of curvature at the common point form a pencil, the locus of centers of curvature thus being a straight line. In the present paper it is shown that this property belongs to a very general class of representations which includes the conformal class as a special case. The representation is next required to have an additional property, also possessed by conformal representations: the curves on the surface which are depicted by the straight lines $x = \text{const.}$, $y = \text{const.}$, are to form an orthogonal net. It is shown that non-conformal representations having both properties arise when, and only when, the surface considered is of the Liouville type. The paper concludes with a discussion of possible representations of a plane on itself.

13. Dr. Kasner's second paper relates to the motion of a particle in a plane under the action of any force depending only upon the particle's position. As the particle describes a trajectory its velocity in general varies. For each set of values of $x, y, y' v$, a unique trajectory is defined, and therefore, at the point in question, a unique value of the curvature or, what is equivalent, of y''. It is thus possible to express the speed v in terms of x, y, y', y''. If now v is replaced by a constant, we have a differential equation of the second order. The curves satisfying this equation are termed "velocity curves." To each speed there corresponds a doubly infinite system of these curves. The first part of the paper investigates these systems, both in general and for conservative, solenoidal, and Laplacian forces. Relations arise to the theory of geodesics, and to isogonal trajectories.

The second part of the paper deals with the total system of ∞^3 velocity curves, corresponding to all values of v. Such a system can never be the system of trajectories either of the same force or of another force. Many of the geometric properties obtained are analogous to the properties of systems of trajectories obtained in a previous paper (see *Transactions* for July, 1906). The properties relate to osculating parabolas and hyperosculating circles; the whole set is completely characteristic.

14. Tchebychev considered the problem of determining, if it exists, the polynomial of given degree n which approximates

most closely to given continuous function of the real variable x in a given finite interval. The generalization considered in Professor Young's paper consists in replacing the given function by an arbitrary function $f(x)$ which is required merely to satisfy a certain functional relation $\phi[f(x)]=0$ of a general character. The existence of at least one polynomial of (closest) approximation of degree n with reference to ϕ is proved under very general hypotheses on the function ϕ. Under somewhat less general hypotheses, a certain necessary condition is derived that a polynomial of degree n be a polynomial of approximation; and this condition is shown also to be sufficient provided a certain determinant does not vanish. With this same restriction, it is shown that the solution is unique, *i. e.*, that there exists only one polynomial of approximation of degree n with reference to a given ϕ.

15. Professor Keyser's paper considers the question: What, if any, analytic fact corresponds to the universal conviction of "natural" man that the points of (our) space constitute one, a whole—(our) space? The hypothesis is suggested as such that any continuous space ${}_nS_e$ of dimensionality n and of elements e of a kind is constituted a space by virtue of some relation r (different for different S's) subsisting between each e and an entity E (different for different S's) itself not an element of (not *in*) ${}_nS_e$. The hypothesis, naturally not admitting of mathematical demonstration, is nevertheless indicated independently by an infinitude of facts and is not inconsistent with any fact of which the writer is aware. Of such indications the following one may be cited as being at once simple and typical. Denote by S_e the ensemble of spheres orthogonal to a sphere S. The elements of S_e are spheres. Let M be an inhabitant of S_e and be conformed to S_e as the intuition of a human intelligence H is conformed to S_p, ordinary point space. Suppose M and H to write each his own geometry and by chance to employ throughout the same nomenclature. Suppose their works exchanged. Then each may read his own geometry in the other's book. M is certain that his elements (spheres) constitute space. H has precisely the same conviction (feeling) with respect to his elements (points). Neither M nor H perceives the ground of his own conviction. H, however, perceives the (at least a) reason for M's conviction. It is that each of M's elements bears a certain same relation r (orthogonality) to a

certain entity E (sphere S) not in M's space S_r. Reciprocity demands a corresponding perception by M in the case of S_p. Like analogies abound. To an inhabitant of a space, to an intelligence of intuition conformed to it, the space as an entity could not appear, as for example a plane dweller could not behold the plane. In case of S_p, E is probably S_p itself as the latter would appear to a beholder outside; and r probably is distantial; specifically, the distance zero of each point from E, and so of the form

$$(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5) : \sqrt{a_1^2 + \cdots + a_4^2} = 0.$$

16. The paper of Dr. Haskins calls attention to an error in the recent paper of Professor Forsyth on "Differential invariants of a plane and of curves in the plane." Professor Forsyth states: (1) that there are p^2 algebraically independent absolute invariants of orders not exceeding p; (2) that these invariants are expressible in terms of certain p^2 geometric magnitudes; and (3) that consequently no relation common to all curves in the plane can exist among these geometric magnitudes. — The error in these statements is due to the fact that the differential form under consideration is of class zero, and hence has no gaussian invariants. There are therefore fewer than p^2 absolute invariants, and hence there exist relations among the p^2 geometric magnitudes mentioned. One such relation, of the third order, is readily established.

17. In Mr. Colpitts's paper, quintic curves lying on a hyperboloid were depicted upon a conic, and the bitangents, inflexions, etc., defined in terms of quartic involution of the quadrisecants. The rational form not lying on a quadric was discussed by means of the octic which fixes the points of contact of the stationary planes. The elliptic form was depicted into a plane curve of order 10, and the surface of trisecants was shown to belong to a linear complex. The form of genus 2 was also treated by means of the plane depiction. Finally, an exhaustive classification of the forms belonging to a linear complex was added.

18. In Mr. Breuke's paper the problem of finding the derivative of the function represented by a series of the form

$$(1) \qquad \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

was considered; and it was shown that if h is a positive integer and α a constant, and if the series

$$(2)\quad \sum_{1-h}^{\infty} \{(n-h)a_{n-h}\cos(nx-h\alpha)-(n+h)a_{n+h}\cos(nx+h\alpha) + (n-h)b_{n-h}\sin(nx-h\alpha)-(n+h)b_{n+h}\sin(nx+h\alpha)\},$$

(where the a's and b's whose subscripts are not positive are all zero) converges uniformly throughout an interval which does not include or reach up to a root of $\sin h(x-\alpha)$, and if (1) converges at some point of this interval, and if as n becomes infinite $\lim a_n = \lim b_n = 0$, then (1) converges uniformly throughout the interval in question, and the function represented by (1) has at every point of this interval a finite derivative which is equal to the value of (2) divided by $2\sin h(x-\alpha)$. If $\alpha = 0$, $h = 1$, this reduces to a theorem of Lerch. Applications of the theorem to the summation of certain trigonometric series were indicated.

19. The purpose of Dr. Rabinovitch's paper is to lay down a set of necessary and sufficient kinetic axioms of geometry, and to prove their mutual independence and compatibility. The primary elements to start with are bodies and their places, and use is made of an auxiliary postulate of the continuity of time regarded as an aggregate of moments.

The axioms are 7 in number.

1) Divisibility.—Each body is divisible into two bodies.

2) Impenetrability.—Each body or each part occupies a place to the exclusion of other bodies or parts.

3) Motion.— A body can have different places at different times. Motion is proved to be a continuous process depending upon time.

By definition two bodies are contiguous when either can enter in part the region of the other during an infinitesimal interval of time, and a body is termed continuous when each of its parts is contiguous to some other parts of the same body.

Axioms 4) and 5) postulate the existence of continuous bodies termed rigid bodies or solids, *i. e.* such that, 1° any two contiguous parts remain contiguous throughout all their motions, and 2° to each particular place of any given part of such a body there corresponds a unique place for every other given part.

6) Homogeneity.—A part can be separated from a given solid, capable of being wholly placed within the space of any other given solid.

7) Axioms of completeness.—No motion is impossible unless it contradicts the above axioms.

From these are deduced the notions of surface, curve, and point. The places which can be occupied by the same solid, its surface or curve are termed congruent. The sphere, circle, straight line, plane, and angle are deduced from the notion of distance, which is a relation of all congruent couples of points. The socalled parallel postulate is deduced by proving that a certain continuous motion of a figure termed "immaterial quadrilateral" cannot contradict the above axioms.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE nineteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 14, 1906. The total attendance was forty, including the following members of the Society:

Dr. L. D. Ames, Mr. G. D. Birkhoff, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Dr. E. L. Dodd, Mr. E. B. Escott, Dr. Peter Field, Professor G. W. Greenwood, Professor A. G. Hall, Professor E. R. Hedrick, Professor T. F. Holgate, Mr. Louis Ingold, Professor O. D. Kellogg, Dr. H. G. Keppel, Mr. N. J. Lennes, Professor H. Maschke, Professor E. H. Moore, Professor F. R. Moulton, Dr. L. T. Neikirk, Professor H. L. Rietz, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor H. E. Slaught, Professor E. J. Townsend, Dr. W. D. Westfall, Professor D. T. Wilson, Mr. R. E. Wilson, Professor Alexander Ziwet.

The chairman of the Section, Professor Alexander Ziwet, presided at the two sessions. Owing to the large number of papers on the programme, it was voted to reduce the time allotments by twenty-five per cent. It was also voted hereafter to restrict the printed programme to those papers for which titles and abstracts are in the hands of the Secretary on the date specified in the preliminary call for the meeting, and to request this notice to be made in the BULLETIN in connection with the announcement of meetings of the Section.

A report of progress was made by the Executive Committee concerning the proposal to devote one or more sessions of the December, 1906, meeting to the consideration of mathematics for engineers, and to invite representatives of engineering schools to present papers and to take part in the discussion.

A resolution was introduced by Professor E. H. Moore and unanimously carried, expressing the very earnest hope of the Chicago Section that it may be found possible to establish a strong section of the Society which shall hold meetings at some convenient center in the Southwest.

The following papers were read:

(1) Professor J. B. SHAW: "Significance of the term hypercomplex number."

(2) Professor J. B. SHAW: "Mathematical processes."

(3) Mr. LOUIS INGOLD: "A theorem on the intersections of Pascal lines."

(4) Professor H. F. BLICHFELDT: "On modular groups isomorphic with a given linear group."

(5) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups" (supplementary paper).

(6) Mr. A. R. SCHWEITZER: "On the foundations of abstract geometry" (preliminary report).

(7) Professor M. E. GRABER: "On the mathematical character of space intuition."

(8) Dr. J. C. MOREHEAD: "Note on the factors of Fermat's numbers."

(9) Mr. N. J. LENNES: "On functions of limited variation."

(10) Professor E. R. HEDRICK: "On the function $\xi(h)$ in the law of the mean."

(11) Professor F. R. MOULTON: "On the classes of periodic orbits computed by Professor G. H. Darwin."

(12) Mr. W. D. MACMILLAN: "On a certain type of periodic orbits in the problem of three bodies."

(13) Mr. F. L. GRIFFIN: "Certain periodic orbits of n finite bodies revolving about a relatively large central mass."

(14) Mr. W. R. LONGLEY: "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies."

(15) Professor H. L. RIETZ: "On normal correlation surfaces."

(16) Professor L. E. DICKSON: "Linear algebras in which division is always uniquely possible."

(17) Dr. E. L. DODD: "An application of Gibbs's exponential

dyadic function e^{ϕ} to the solution of linear vector and dyadic differential equations."

(18) Dr. E. L. DODD: "An example preparatory to the study of the ratio test for infinite series.

(19) Mr. R. E. WILSON: "On integral equations of the second kind."

(20) Mr. R. E. WILSON: "On integral equations of the first kind."

(21) Professor D. R. CURTISS: "A proof of the theorem concerning artificial singularities."

(22) Professor D. R. CURTISS: "On certain properties of wronskians and related matrices."

(23) Professor HEINRICH MASCHKE: "On spherical images of rectilinear congruences" (preliminary report).

(24) Dr. W. D. WESTFALL: "On integral equations of the second kind."

(25) Professor O. D. KELLOGG: "The roots of transcendental integral functions."

The papers of Professor Blichfeldt, Professor Graber, and Professor Maschke were read by title. Mr. MacMillan, Mr. Griffin, and Mr. Longley were introduced by Professor F. R. Moulton. Abstracts of the papers follow below, the numbers corresponding to the titles in the list above.

1. Professor Shaw's first paper compares critically the definitions of a hypercomplex number from the four standpoints of mutiplex, n-dimensional number, operator, and extension of ordinary number. The first definition is exemplified in the definition of the complex number $a + b\sqrt{-1}$ as a couple (a, b), where a and b are positive or negative reals, subject to the law of multiplication $(a, b)(c, d) = (ac - bd, ad + bc)$. The second is exemplified in the definition of $a + b\sqrt{-1}$ as a two dimensional number $ae_0 + be_1$, where $e_0^2 = e_0$, $e_1^2 = -e_0$, $e_0e_1 = e_1e_0 = e_1$. The third is exemplified in the definition of $a + b\sqrt{-1}$ as an operator which rotates through an angle $\tan^{-1} b/a$ and multiplies by $\sqrt{a^2 + b^2}$. The fourth defines $a + b\sqrt{-1}$ as a value of x which satisfies the ordinary equation $x^2 - 2ax + d = 0$, where $d = a^2 + b^2$. The latter algebraic definition is considered to be the most general since it does not require that from the equality of two hypercomplex numbers $a = b$ we must conclude $a_i = b_i$ $(i = 1, \cdots, n)$, where a_i is the i-th coordinate

of the number. This view permits us to call, for example, the algebra of positives and negatives a double or two-unit (qualitative unit) algebra, which the other definitions do not permit.

2. In his second paper Professor Shaw defined a mathematical process to be a definite procedure by which from given data we pass to a certain conclusion. The question of validity, objective or subjective, of the process is handed over to philosophy. The theory of mathematical processes bears to logic the relation that modern rational geometry has to euclidean geometry. It is also a generalized theory of mathematical forms. For example, on one hand the syllogism is exhibited as one of an infinite variety of modes of reasoning; on the other hand the theory of laws of combination, like the commutative law, is reduced to a problem in isomorphisms of substitution groups. This theory appears to be the common foundation of mathematics and logic. This paper is to be presented to the *Transactions* for publication.

3. In Mr. Ingold's paper are considered some properties of the configuration $\left(\begin{smallmatrix}12 & 3\\ 4 & 16\end{smallmatrix}\right)$ in their relation to a conic through six of its points. From these properties are immediately deducible a large number of results as to collinearity and concurrence of points and lines connected with the inscribed hexagon and inscribed (any) pentagon. In particular it is shown that certain joins (other than Pascal lines) of diagonal points of the hexagon meet by sixes in sixty points which lie by fours on the sides of the hexagon and by threes on twenty other lines. The results applied to a reciprocal situation show that certain of the intersections of Pascal lines (Cayley's 360 points, *Quarterly Journal*, volume 9) lie by sixes on sixty lines which pass by tens through the vertices of the hexagon.

4. In his first paper Professor Blichfeldt proves the following theorem: Given a group G of linear homogeneous substitutions in n variables, transitive (irreducible) and of finite order, then there exists an infinitude of prime numbers p for which we can construct an isomorphic transitive group G' of linear homogeneous substitutions in n variables, the elements of whose matrices are integers taken modulo p.

5. Professor Blichfeldt's paper on the order of linear homogeneous groups is supplementary to his two former papers in

the *Transactions*. He reduces his former superior limit $(n-1)(2n+1)$ of the value of a prime p which may divide the order of a finite primitive group of linear homogeneous substitutions of determinant unity in n variables. If such a group has a substitution of order p $(p>n)$ and variety m $(m \leqq n)$, but none of order pk, then $p \leqq 3m+n-3$. The primes which may divide the orders of the primitive collineation groups in 4, 5 and 6 variables are not greater than 11, 13 and 19, respectively. The two papers by Professor Blichfeldt will appear in the July number of the *Transactions*.

6. Mr. Schweitzer sets up systems of axioms for n dimensions $(n=1, 2, 3, \cdots)$. Two undefined symbols enter the theory for n dimensions, viz., points as elements and a relation K, effective under conditions specified in the postulates, between two ordered n-ads of points. The postulates are laid down so as to permit of ready definition of betweenness, collinearity, etc. Systems III, etc., are shown to be sufficient for projective geometry. System I is easily extensible to higher dimensions on comparison with systems II, III, $\cdots$. The extensions of the remaining systems are quite obvious. Such extensions are, however, less elegant than the corresponding sets in the first systems.

7. In this paper Professor Graber develops the theory that space intuitions do not change in nature, but in number and complexity. That in the naïve intuition of space the intuitional elements are few and their complexes restricted within narrow limits, but as geometric knowledge increases, the number of intuitions and intuitional complexes increases, and we have consequently more exact knowledge of spatial relations.

In the intuitional complexes formed by combinations of intuitions a, b, n, viz., $\Sigma(a, b, c, \cdots, n)$, we may have (1) annihilators, *e. g.*, $\widehat{ab} \equiv 0$; (2) correctors, *e. g.*, $\widehat{ab} = a, b$, or a complex partaking of the nature of a and b; (3) transmutants, *e. g.*, $\widehat{ab}$ = a complex indicating a distinct advance.

It is then attempted to show that the ensemble of our space intuitions sustains a functional relationship to geometric progress, the intuitions leading to geometric research and vice versa. From this it would follow that no geometric problem is completely solved until we are able to interpret both the method of solution and the results in terms of the intuition.

8. Dr. Morehead's paper appears in full in the present number of the BULLETIN.

9. The variation of a function $f(x)$ on an interval ab is denoted by $V_a^b f(x)$. Mr. Lennes's paper is concerned with the variation of the definite integral regarded as a function of its limits of integration. It is known that $\int_a^x f(x)dx$ is a function of limited variation on an interval ab on which $f(x)$ is properly integrable. It is shown that

$$V_a^b \int_a^x f(x)dx = V_a^b \int_a^x |f(x)|\,dx.$$

In case the integral $\int_a^b f(x)dx$ exists improperly it is shown that the existence of $\int_a^b |f(x)|\,dx$ is a necessary and sufficient condition that $\int_a^x f(x)dx$ shall be a function of limited variation on ab. A new proof is given of a theorem proved by Jordan and Stolz, and announced by Moore with reference to these authors (*Transactions*, volume 2, page 322), viz., that the existence of the (Moore's) broad integral is sufficient for the existence of $\int_a^b |f(x)|\,dx$ either narrowly or broadly.

10. In this paper Professor Hedrick develops some of the properties of the function $\xi(h)$ in the law of the mean, written in the form

$$\frac{f(a+h)-f(a)}{h} = f'(a+\xi).$$

In particular it is shown that this law holds in a slightly modified form even in case the derivative $f'(x)$ exists merely with regard to certain assemblages. The possible discontinuities of $\xi(h)$ and of $f'(x)$ are then discussed. It is shown that $\xi(h)$ is always continuous at $h=0$ with respect to an assemblage which has the power of the continuum in any interval about $h=0$; and again that $f'(x)$ is not necessarily continuous for the entire assemblage of even those values which ξ actually assumes.

Finally, remarks concerning the general nature of a derivative, and concerning the necessary and sufficient conditions for a minimum of $f(x)$ close the paper. The paper will be presented to the *Annals of Mathematics*.

11. The paper of Professor Moulton consists of an application of the analysis which he presented to the Society at the summer meeting, 1905, to the periodic orbits discussed by Professor G. H. Darwin in *Acta Mathematica*, volume 21. The satellites of class A are of the type of the orbit of the moon when its inclination and eccentricity are neglected. A numerical example agreeing with Darwin's figures is given. The orbits of classes B and C are the analytic continuations of orbits whose coordinates are conjugate complex numbers for vanishing values of the disturbance. When expanded as power series in the lunar theory parameter m, the expressions for the coordinates of the orbits A, B and C all have the same radius of convergence. These series cease to converge for the value of m for which B and C become real; and this value of m probably gives their true radius of convergence. Numerical application is also made to planets A. The oscillating satellites were treated in an earlier paper.

12. Mr. MacMillan treats in his paper a certain type of periodic orbits of an infinitesimal body under the attraction of two finite bodies which revolve about one another in ellipses. The existence of periodic orbits of this type in the plane is demonstrated and a convenient method is given for constructing the solutions as power series in a parameter with periodic coefficients.

13. In this paper Mr. Griffin discusses a certain class of periodic solutions of the problem of n bodies where one of the masses is very large with respect to the others. In other words the distribution of the masses is such as is presented by the sun and any number of planets, or by a planet and any number of satellites. The bodies move in a plane, with their undisturbed synodic mean motions commensurable. Using the differential equations in all their generality, the existence of periodic orbits under the mutual disturbances is proved, the demonstration being particularly simple when all the bodies start from a "symmetric conjunction." A method of constructing the solutions as power series in the ratio of one of the

small masses to the large one is given in detail, the solution being unique when the period, the epoch, and the longitude of the initial conjunction have been selected. As an application, numerical work is given for the case of Jupiter's satellites, I, II, III. The fact that under certain circumstances there are no periodic orbits seems to offer information as to the reason for the existence of the "lacunary spaces" in the system of asteroids, and in Saturn's rings.

14. Mr. Longley assumes that the coordinates of the n finite bodies are known periodic functions of the time; for example, the generalized lagrangian solutions. The orbits considered are those of a particle of infinitesimal mass which revolves about one of the finite bodies. The problem is restricted to one plane, and the differential equations of motion are treated rigorously, the solutions being expressed as power series in parameters which converge when they are sufficiently small. It is shown that when a period (necessarily a multiple of the period of the finite bodies) is preassigned, there exist two and only two orbits of the particle, of the type considered, which are periodic with the prescribed period. The motion in one is direct and in the other retrograde. A convenient method for constructing the solutions is given, and some special cases are considered in detail.

15. Professor Rietz gives a derivation of correlation surfaces based upon the nature of certain plane sections of the surface. In the derivations hitherto given, the form of the function which defines the surface has either been assumed, or justified by assumptions as to the distribution and independence of the causes which produce deviations from the mean value. Then the constants in this function are determined. Much use has been made of correlation surfaces in recent years for the purpose of giving mathematical expression to facts of organic evolution. From the manner in which the applications are made, the nature of plane sections parallel to the two coordinate planes is of primary importance. It therefore seems well to make this the starting point of a derivation of these surfaces. It is shown in this paper how correlation surfaces which describe many natural phenomena can be obtained from certain types of sections.

16. Professor Dickson considers algebras whose elements are

$\Sigma a_i e_i$, the m coördinates a_i ranging over an arbitrarily given field F. For $m = 3$, he considers the most general linear transformation of the units and exhibits families of algebras invariant under every linear transformation. The algebras which admit more than one transformation (and hence exactly three transformations) into themselves are completely determined. From each of these standpoints he is led to the same remarkable set of families of algebras, each set characterized by a parameter μ. For $\mu = 1$, the family is the system of all fields of rank 3 with respect to F. For $\mu = 0$, the family is the system of algebras

$$i^2 = j, \quad ij = ji = b + \beta i + Bj, \quad j^2 = 4bB - \beta^2 - 8bi - 2\beta j,$$

where b, β, B range over the marks of F (assumed not to have modulus 2) for which $x^3 - b - \beta x - Bx^2$ is irreducible; in such an algebra division is always uniquely possible. For m an even integer $\geqq 4$ and F an arbitrary field, there exists a very interesting type of algebra in which division is always possible. For $m = 4$, the algebra is given by

$$i^2 = j, \quad ij = ji = k, \quad ik = ki = c, \quad j^2 = -c + dj, \quad jk = kj = -ci + dk, \quad k^2 = cd - cj \qquad \begin{pmatrix} c \text{ and } d^2 - 4c \\ \text{not squares in } F \end{pmatrix}.$$

In fact, its determinant equals

$$(x^2 + dxy + cy^2)^2 - c(z^2 + dzw + cw^2)^2,$$

which vanishes only when x, y, z, w all vanish.

While there exists a large number of non-equivalent m-tuple algebras in a given field (for example, if $m = 3$ and F is the field of order 3 or 5, the number is 6 or 38 respectively), there appears to be a single non-field commutative algebra for each field when $m = 3$ or 4. However, for $m = 6$ there exist two non-equivalent, non-field commutative algebras in every field.

A general investigation was made of linear associative algebras in which division is always uniquely possible. An interesting example of such an algebra is one with the n^2 units $i_0^r e^s$ $(r, s = 0, 1, \cdots, n-1)$, where $e^n = \tau =$ rational function of i_0, and $e^s i_0^r = i_s^r e^s$, while $\imath_0, i_1 = \theta(i_0), i_2 = \theta^2(i_0), \cdots$ are the roots of a uniserial abelian equation of degree n irreducible in F. Further, the coefficients of the latter and τ must satisfy certain conditions which can be expressed in terms of the theory of homogeneous forms of degree n. The paper will appear in the *Transactions*.

17. Dr. Dodd's first paper is in abstract as follows: Gibbs defines the direct product $\alpha \cdot \beta$ of two vectors as the product of their lengths into the cosine of the included angle. A dyad $\gamma\delta$ is an operator transforming one vector into another. Thus $(\gamma\delta)\cdot\alpha = (\delta\cdot\alpha)\gamma$. The product of two dyads is a dyad, *e. g.*, $(\gamma\delta)\cdot(\epsilon\zeta) = (\delta\cdot\epsilon)(\gamma\zeta)$. A dyadic, Φ, is a sum of dyads, and may be expressed in nonion form, thus:

$$\Phi = a_{11}ii + a_{12}ij + a_{13}ik + a_{21}ji + a_{22}jj + a_{23}jk + a_{31}ki + a_{32}kj + a_{33}kk,$$

where i, j, k, are unit vectors at right angles, and the a's are scalars. In the product, $\Phi\cdot\Psi$, multiplication is distributive and, if Φ and Ψ are in nonion form, follows the row-column rule for the multiplication of determinants. $\Phi^2 \equiv \Phi\cdot\Phi$. The idem-factor $I = ii + jj + kk$ is such that $I\cdot\rho = \rho$ and $\rho\cdot I = \rho$, where ρ is the vector. Finally Gibbs defines

$$e^{\Phi} = I + \Phi + \frac{1}{2!}\Phi^2 + \frac{1}{3!}\Phi^3 + \cdots$$

We find that, in general, $e^{\Phi+\Psi} \neq e^{\Phi}\cdot e^{\Psi}$, but if t and Δt are scalars, *i. e.*, numbers, real or complex, then $e^{(t+\Delta t)\Phi} = e^{t\Phi}\cdot e^{\Delta t\Phi}$; so that

$$\frac{d}{dt}e^{t\Phi} = \Phi\cdot e^{t\Phi} = e^{t\Phi}\cdot\Phi.$$

Then, if ρ is a vector and $\Gamma_0, \Gamma_1, \cdots, \Gamma_n$ constant dyadics, a solution of

$$(1)\qquad \Gamma_n\cdot\frac{d^n\rho}{dt^n} + \Gamma_{n-1}\cdot\frac{d^{n-1}\rho}{dt^{n-1}} + \cdots + \Gamma_1\cdot\frac{d\rho}{dt} + \Gamma_0\cdot\rho = 0$$

will be $\rho = e^{t\Phi}\cdot\alpha$, if α is an arbitrary vector, and Φ is a solution of

$$\Gamma_n\cdot\Phi^n + \Gamma_{n-1}\cdot\Phi^{n-1} + \cdots + \Gamma_1\Phi + \Gamma_0 = 0.$$

In (1), ρ may be replaced by a dyadic Ψ if a dyadic Ω replaces α. If in (1) $\Gamma_n = c_n I$, etc., where the c's are scalar constants, then (1) reduces to

$$c_n\frac{d^n\rho}{dt^n}+c_{n-1}\frac{d^{n-1}\rho}{dt^{n-1}}+\cdots+c_1\frac{d\rho}{dt}+c_0=$$

Here substitute $\rho=e^{mtI}\cdot\alpha=e^{mt}\alpha$, where m is a scalar. Cf. Grassmann's Ausdehnungslehre, § 498.

18. The example of ratio test devised by Dr. Dodd is the following:

Let $$S=u_1+u_2+u_3+\cdots+u_n+\cdots$$

$$=\frac{1}{5}+\frac{1\cdot3}{5\cdot10}+\frac{1\cdot3\cdot5}{5\cdot10\cdot15}+\cdots+\frac{1\cdot3\cdots(2n-1)}{5\cdot10\cdots5n}+\cdots.$$

Then $$r=\frac{u_{n+1}}{u_n}=\frac{2n+1}{5(n+1)},\qquad \lim_{n=\infty} r=\frac{2}{5},\quad r<\frac{2}{5}.$$

The sum of S exists, and is less than 0.2912, by comparison with

$$S'=u_1+u_2+u_3+u_4+u_5+(2/5)u_5+(2/5)^2u_5+(2/5)^3u_5+\cdots.$$

The sum of S is greater than .2909, by comparison with

$$S''=u_1+u_2+u_3+u_4+u_5+(11/30)u_5+(11/30)^2u_5+\cdots.$$

Then $S=0.291$ is correct to within 0.0002.

An infinite series whose test ratio approaches a limit l will ultimately become approximately a geometric series with ratio l, and it will converge if $|l|<1$.

19. Mr. Wilson's first paper treats of the reduction of a differential equation of the third order under certain boundary conditions to an integral equation of the second kind, and a generalization to differential equations of order $2n-1$. After showing that the matrix of such an integral equation is always skew-symmetric, he applies his results to the differential equation $y'''-4py'-2p'y=0$, which arises in the solution of Lamé's equation $y''=(Ax+B)y$ by the application of a method due to Hermite, and shows that for a fixed value of A there exists an infinity of normal solutions which satisfy the given boundary

conditions. He further shows under what conditions an arbitrary function is developable in terms of these normal solutions.

20. In his second paper Mr. Wilson discusses the representation of arbitrary functions by means of Fourier's double integrals and shows the relation of the theory of integral equations of the first kind to the theory of Fourier's integrals. Incidentally he gives a few examples of integral equations of the first kind which are readily solved and which in themselves might serve as the basis of Fourier's theory.

21. The first paper of Professor Curtiss gives a proof of the fundamental theorem which states that if $f(z)$ is single valued and analytic at all points in the neighborhood of a point c, exclusive of c, and remains finite throughout this neighborhood, it has at most an artificial singularity at c; *i. e.*, with a suitable definition of $f(c)$ the function $f(z)$ will be analytic at the point c. In some fallacious proofs use has been made of the auxiliary function

$$\phi(z) = (z - c)f(z) \quad (z \neq c),\ \phi(c) = 0.$$

A very simple demonstration is here given, based on the consideration of the function $(z - c)\phi(z)$. This paper will be presented to the *Annals of Mathematics* for publication.

22. The second paper of Professor Curtiss will appear in the July number of the BULLETIN.

23. Professor Maschke discussed the solution of the general problem of the determination of rectilinear congruences whose spherical images are given, *i. e.*, the determination of x, y, z in terms of X, Y, Z and the quantities e, f, f', g defined by

$$\Sigma \frac{\partial X}{\partial u}\frac{\partial x}{\partial u} = e, \quad \Sigma \frac{\partial X}{\partial u}\frac{\partial x}{\partial v} = f, \quad \Sigma \frac{\partial X}{\partial v}\frac{\partial x}{\partial u} = f', \quad \Sigma \frac{\partial X}{\partial v}\frac{\partial x}{\partial v} = g.$$

24. In this paper Dr. Westfall shows that the restriction that the matrix $K(s, t)$ be symmetric in s, t in Hilbert's treatment of the integral equation of the second kind

$$f(s) = \phi(s) - \lambda \int_0^1 K(s, t)\phi(t)dt$$

is not necessary for that treatment. A slight change in his proof gives the relations

$$K(s, t) + \mathrm{K}(s, t) = \lambda \int_0^1 K(s, r)\mathrm{K}(r, t)dr,$$

$$K(s, t) + \mathrm{K}(s, t) = \lambda \int_0^1 \mathrm{K}(s, r)K(r, t)dr,$$

which prove the existence and uniqueness of the solution

$$\phi(s) = f(s) - \lambda \int_0^1 K(s, t)f(t)dt.$$

25. The fact that the roots of an integral algebraic function are continuous functions of the coefficients may be generalized to transcendental functions, and the result very simply applied to give certain information concerning the roots of the latter. Professor Kellogg proposes two applications of this notion, the first in building up a transcendental integral function term by term, so that it appears that if the convergence of the series is rapid enough, it will surely have finite roots. By a second application the series is considered as a polynomial plus a remainder. If the remainder is sufficiently small all the roots of the polynomial have corresponding roots in the complete function.

H. E. SLAUGHT,
Secretary of the Section.

CHICAGO, ILL.,
April 20, 1906.

GROUPS IN WHICH ALL THE OPERATORS ARE CONTAINED IN A SERIES OF SUBGROUPS SUCH THAT ANY TWO HAVE ONLY IDENTITY IN COMMON.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, April 28, 1906.)

1. WE begin with the case where the group G is any abelian group such that all of its operators are contained in a series of subgroups $H_1, H_2, \cdots, H_\Lambda$ any two of which have only

identity in common. In other words, the λ given subgroups contain every operator of G, except identity, once and only once. It is easy to prove that the order of G is p^m, where p is a prime, and that the type of G is $(1, 1, 1, \cdots)$. If the order of G were divisible by two distinct primes, one of the λ subgroups H_α would involve an operator of some prime order p while another subgroup H_β would involve an operator of a different prime order q. The product of these two operators would be in H_γ, where γ is different from α and β. As the qth power of this product would be in H_α, the two subgroups H_α, H_γ would have an operator of order p in common. Since this is contrary to the hypothesis, it follows that the order of G is p^m.

If one of the λ subgroups H_α contained an operator of order p^2, the product of this operator into an operator of order p from H_β would be in H_γ. As this operator would be of order p^2 and would have its pth power in H_α, it follows that none of the subgroups in question can involve an operator of order p^2 and hence G is of type $(1, 1, 1, \cdots)$. Moreover, whenever G is such an abelian group, its operators may be arranged in subgroups having only identity in common. One such arrangement is effected by letting each of the H's represent a subgroup of order p. In this case $\lambda = (p^m - 1)/(p-1)$.

In what precedes no condition was imposed upon the subgroups $H_1, H_2, \cdots, H_\lambda$ except that they include all the operators of G and that any two of them have only identity in common. If we impose the additional condition that all of these subgroups are of the same order, it follows from the preceding paragraphs that this order is a power of a prime. Moreover if each of these subgroups is of order p^a, $\lambda = (p^m-1)/(p^a-1)$. That is, m is divisible by a. That this condition is sufficient as well as necessary follows from the fact that we may represent G as the direct product of m/a subgroups of order p^a. If these subgroups are represented as regular substitution groups, the λ subgroups may be obtained by arranging these substitution groups in (1, 1) correspondence (taking first two at a time, then three at a time, etc.) and then transforming cyclically $p^a - 1$ elements of a constituent which differ from the identity.*

The main results which have been obtained in the preceding

*Cf. Moore, BULLETIN, vol. 2 (1895), p. 38.

paragraphs may be stated as follows: All the operators of an abelian group cannot be arranged in subgroups such that any two have only identity in common unless the order of the group is p^m and its type is $(1, 1, 1, \cdots)$. The necessary and sufficient condition that all the operators of this abelian group can be arranged in subgroups of the same order p^a, such that any two of these subgroups have only identity in common, is that m is a multiple of a.

2. We shall now consider the case where the group G is non-abelian and of order p^m.

As in the preceding case we do not, at first, impose any conditions upon the orders of $H_1, H_2, \cdots, H_\lambda$. Suppose that at least one of these subgroups H_a contains operators of order p^2. Since every subgroup of a group of order p^m is transformed into itself by operators which are not in it, H_a is invariant under a group of order p^{a+1}, p^i being the order of H_a. This contains no operators of order p^2 except those of H_a, since no two of the H's have operators of order p in common. The operators of order p^2 in H_a generate a characteristic subgroup K under this group of order p^{a+1}. As the latter is invariant under a group of order p^{a-2}, K must also have this property. If this group of order p^{a+2} involved any operators of order p^2 besides those of K, such an operator into an invariant operator of order p in K would give a product of order p^2 which would be in a different H but would generate the same subgroup of order p as the preceding operator of order p^2. As this is contrary to the hypothesis that any two H's have only identity in common, and as the same argument would apply to larger groups if the group of order p^{a+2} did not involve any operators of order p^2 besides those of K, we have proved the theorem: *If a non-abelian group of order p^m is such that all of its operators are found in a series of subgroups of which no two have any common operator except identity, then only one of these subgroups can involve operators of order p^2.*

We impose now the additional condition that the subgroups $H_1, H_2, \cdots, H_\lambda$ have the same order and prove that, in this case, all the operators of the non-abelian group of order p^m are of order p, with the exception of identity. If this were not the case, the operators of order p^2 would generate a characteristic subgroup K contained in H_a. Any other H would transform K into itself and hence it would involve operators of order p which would be commutative with operators of order

p^2 in K. As the product of such operators of order p and p^2 respectively would be of order p^2 but would not be in K, this is impossible. That is, *G cannot involve any operators of order p^2 when the subgroups $H_1, H_2, \cdots, H_\lambda$ have the same order.* In fact, the preceding proof holds when the order of the largest of these subgroups does not exceed p times the order of some other one of them.

The preceding proof can be directly extended so as to apply to any group of any order whatsoever in which all the operators are found in a series of subgroups of the same order such that any two of them have only identity in common. That is, such a group G cannot involve any operator whose order is the square of some number.* Suppose that G involved an operator of order p^2, where p is a prime, and let P represent one of its Sylow subgroups of order p^m. If $P_1, P_2, \cdots, P_\lambda$ represent the subgroups of P which are found in the different subgroups of G which have only identity in common, it follows from what was proved above that not more than one of these subgroups can involve operators of order p^2. The operators whose orders exceed p in P would therefore generate a subgroup of order p^α where α does not exceed $\frac{1}{2}m$. As this is impossible, we have proved that G cannot contain an operator whose order is a square greater than unity.

NOTE ON THE FACTORS OF FERMAT'S NUMBERS.

BY DR. J. C. MOREHEAD.

(Read before the Chicago Section of the American Mathematical Society, April 14, 1906.)

FERMAT'S numbers $F_n = 2^{2^n} + 1$ are known to be prime for $n = 0, 1, 2, 3, 4$, and composite for $n = 5, 6, 7, 9, 11, 12, 18, 23, 36, 38$. By calculating the residues (mod $2^{75}\cdot 5 + 1$) of the reciprocals †

*The minimum order of G is evidently the square of the order of one of these subgroups. Dr. Manning proved that G is abelian whenever it has this minimum order.

†In many cases the residue of $1/2^{2^n}$ (mod N) is more readily calculated than the residue of 2^{2^n}. In the present case $-2^{75}\cdot 5 \equiv 1 \bmod (2^{75}\cdot 5 + 1)$. Therefore $1/2^{2^6} \equiv -2^{11}\cdot 5$, $1/2^{2^7} \equiv 2^{22}\cdot 5^2$, $\cdots$, $1/2^{2^9} \equiv 2^{88}\,5^8 \equiv -2^{13}\cdot 5^7$, $\cdots$, $1/2^{2^{12}} \equiv -5^{26}\cdot 10^{29}$, at which stage division by $2^{75}\cdot 5 + 1$ may be begun.

$$1/2^{25}, \quad 1/2^{27}, \quad 1/2^{29}, \cdots$$

I have found that

$$-1 \equiv 1/2^{273} \pmod{2^{75} \cdot 5 + 1},$$

which establishes the composition of F_{73}.

This leads to an interesting secondary result,—the identification of the *twenty-four place* prime*

$$P = 2^{75} \cdot 5 + 1 = 188\ 894\ 659\ 314\ 785\ 808\ 547\ 841.$$

The prime character of P is a consequence of the fact that all factors of F_{73} must have the form

$$Q = 2^{75} q + 1.$$

For, if P were not prime then a factor of P, $< \sqrt{P}$, and hence not of the form Q, would be a factor of F_{73}.

Further computation recently carried out shows that under $2^{79} \cdot 5 + 1$, the only factors of F_n of the form $2^{\kappa} \cdot 5 + 1$ are

$$2^{7} \cdot 5 + 1, \quad 2^{25} \cdot 5 + 1, \quad 2^{39} \cdot 5 + 1, \quad 2^{75} \cdot 5 + 1;$$

and in fact, for $\kappa < 79$, $2^{\kappa} \cdot 5 + 1$ is prime only for $\kappa = 1, 3, 7, 13, 15, 25, 39, 55, 75$.

An investigation of the numbers $2^{\kappa} \cdot 3 + 1$ shows that $2^{41} \cdot 3 + 1$ is the only number of this form under $2^{105} \cdot 3 + 1$ that is a factor of a Fermat number, the only *odd* values of κ for which $2^{\kappa} \cdot 3 + 1$ is prime being $\kappa = 1, 5, 41$. A prime of the form

$$\Pi = 2^{2\kappa} \cdot 3 + 1$$

cannot be a factor of a Fermat number. For if Π is a factor of F_n, $n \leqq 2\kappa - 2$, the congruence

$$2^{(\Pi-1)/3} \equiv 1 \pmod{\Pi} \tag{1}$$

must be satisfied; and this requires that Π be expressible in the form †

$$\Pi = a^2 + 3b^2 \tag{2}$$

* $2^{61} - 1$ is the highest number hitherto definitely known to be prime. See Seelhoff, *Zeitschrift für Math. u. Phys.*, vol. 31 (1886), p. 178; and Ball, *Messenger*, vol. 21, pp. 34–40.

† For general conditions that $2^{(p-1)/n} \equiv 0 \pmod{p}$, see Cunningham, *Quar. Jour. of Math.*, vol. 37 (1905), pp. 124–135.

where

$$b \equiv 0 \pmod 3. \tag{3}$$

Since a given prime is expressible in not more than one way in the form $a^2 + 3b^2$, and since

$$\Pi = (1)^2 + 3(2^\kappa)^2$$

it follows that condition (3), and therefore (1), is not satisfied.

It is easy to show that *composite* numbers of the forms $2^{\kappa \cdot}3 + 1$, $2^{\kappa \cdot}5 + 1$ can not be factors of Fermat's numbers.

NORTHWESTERN UNIVERSITY,
April, 1906.

THEORETICAL MECHANICS.

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies. By E. T. WHITTAKER. Cambridge, The University Press; New York, The Macmillan Company, 1904. xiii+414 pp.

AT Cambridge, England, mathematics means for the most part mechanics, mathematical physics, or even physics sometimes not so very mathematical. The famous tripos — the mathematical tripos, of course, which goes back at any rate to 1747—seems, at least to an outsider, to lay its main stress on the theoretical application of mathematics rather than on pure mathematics. Very likely this is a tradition that has come down from the time of Newton, and it is certainly maintained by the eminent physicists such as Stokes, Kelvin, Maxwell, Rayleigh, J. J. Thomson, who have been high wranglers. This tripos with its great prestige gives an attractive and distinctive touch to the university and although the ever-increasing pressure of pure mathematics, with its possibilities for various kinds of unessential and mediocre work infinitely wider than those to be found in theoretical applied mathematics, will probably tell on the training at Cambridge sooner or later, may we not look forward to that date with some regrets on the general uniformizing that is taking place and lament the fact that other realms than physics are possessed of an entropy? At present, how-

ever, there seems no immediate danger. The one year of 1904 saw the publication at the Cambridge University Press of three large and highly important volumes by as many recent graduates of Cambridge typically cantabrigian in that they exhibit great mathematical power and attainments directed firmly and unerringly along the direction of physical research. It is needless to say that we refer to Walker's Analytical theory of light, Jean's Dynamical theory of gases, and Whittaker's Analytical dynamics.

Mechanics is more largely pure mathematics than the other branches of theoretical physics because the data are fewer and simpler; the transition to the systems of differential equations, less recondite. And so far is mechanics developed that probably it is the mathematician who must carry on the work rather than the physicist engrossed in advancing the experimental data of science—witness the recent workers, Poincaré, Painlevé, Hadamard, Duhem, Volterra, Levi-Cività, Gibbs, Love, Jeans, Klein, and others. For the mathematician, however, the way has unfortunately been barricaded by the kind of treatise on mechanics which has been available to the intermediate student. The famous works of Routh are not so much mathematical as distinctly mechanical; the reader learns rather how to do difficult and often artificial problems rather than how problems are done. This training is highly valuable and should precede other work: but where to find the other? It is precisely at this point that Whittaker breaks the barricade and opens the way to fruitful advance. His book, to be sure, starts at the beginning of the subject—but it is not for the beginner. A good course from the practical problem-solving aspect of Routh is almost indispensable to a ready and successful mastery of Whittaker. The book is mathematical in nature, written with a precision and developed with a logic sure to appeal to mathematicians. It is modern, thoroughly modern with its bibliographical references running quite up into the year 1904. Its subjects include the contributions of "Lie, Rayleigh, Klein, Hertz, Lorentz, Poincaré, Siacci, Bruns, Boltzmann, Larmor, Greenhill, Appell, Painlevé, Stäckel and Levi-Civita" to quote from the preface. And this is quite true. Those who are familiar with a number of treatises on mechanics may scan them in vain for a résumé of the methods introduced since 1880 just as they may look equally in vain for an advanced book arranged in what seems a logical order from the mathematician's point of view.

It is therefore a definite and strategic position which Whittaker has occupied and one may in this instance put aside all lamentations from the observation that of the making of books there is no end. To show this, needs but a detailed examination of the contents. First, however, it may be well to mention that the methods of attack which the author displays throughout the text are numerous. Be they geometric or algebraic or analytic he invariably selects that which moves most directly to the goal of solution. This diversity of method taken with the compact style makes the book hard reading for any but the somewhat advanced student. To solve the exercises requires on the part of the reader a similar versatility. Thus one may see why and how it is that the Cambridge student well versed in mechanics is also in reality thoroughly equipped in pure mathematics in so far as present practical applications are concerned. To a greater extent than his predecessors Whittaker has availed himself of the great machinery of modern mathematics, has thrown life into both the mathematics and the mechanics which he treats, and in this way has produced a book which may well become an inspiration to the next generation as Routh's treatises have been to this.

The first chapter of the text is on kinematic problems. Here, condensed into twenty-five pages, may be found Euler's, Hamilton's and Chasles's theorems on displacements, with their applications to instantaneous axes and centers; also the various methods of representing a displacement analytically, whether by the angle of version, the coördinates of Rodrigues, the angles of Euler, or the parameters of Klein, with the formulas for transformation from one to another and the expressions for the velocity and accelerations in the different systems of coördinates. It should be stated that the chapter further contains about twenty-five problems of all sorts and grades for the reader to work. This is a feature of the book. The problems cannot all be equally well selected and equally instructive. A large number of them are branded "(Coll. Exam.)" and frequently they would have shown it only too obviously without the mark. On the other hand many are taken from published memoirs of prominent writers and are excellent, though often difficult. The solution of a considerable number of problems chosen throughout the work indicates that the answers are usually correct. The second chapter is on the equations of motion and is of equal length—or rather brevity. It does for dynamics what

the first did for kinematics. From the simple newtonian equations for the motion of a particle the author proceeds by rapid strides to the ideas of work and of generalized coördinates, to the distinction between holonomic and non-holonomic systems, and to the lagrangian equations for holonomic systems, to conservative systems, kinetic potential, quasi coördinates, and finally to impulsive motion. It is needless to point out that for one not already somewhat familiar with the subject these twenty-five pages must be taken in small doses mixed with similar portions taken from the fourth and later chapters. This second chapter is purely on the theory and not on the practice of the equations of dynamics.

In the discussion of the principles available for integration, which makes up Chapter III, the author continues his mathematical investigations in much the same style. The reader will again do well to intersperse some parts of the later chapters with his reading. With this and the ten or a dozen problems given in immediate connection with the investigations there is no very great difficulty to impede one's progress. After a word about what is meant by the solution of a system of differential equations, the question of ignorable coördinates is taken up in detail, and the theorem is proved that a dynamical system with n degrees of freedom which has k ignorable coördinates can be reduced to a dynamical system which has only $n - k$ degrees of freedom. And this possibility of ignoring some coördinates turns out to be the most general reason for the actual solvability of some problems and non-solvability of others. Too much emphasis cannot be laid upon this principle of the ignoration of coördinates, for it gives a definite and regular method of simplifying the problem, and frequently it alone may be made to yield the desired information about a system the actual complete solution of which we are either unable to obtain or able to obtain only by the expenditure of too much effort. Simple examples of the principle are found in systems possessing one or more integrals of momentum or of angular momentum. In this connection it should be noted that the kinetic energy of the ignored coördinates appears as potential energy in the modified system. This has given rise to a question which we are not yet in a position to answer satisfactorily, namely whether or not all potential energy may be explained as kinetic energy of some ignored coördinates, and how? It is obvious to all, that potential energy is to a considerable extent merely a convenient name

for the manifestations of something the mechanism of which we do not understand and frequently "explain" by action at a distance. This instructive chapter closes with a discussion of the integral of energy in a conservative system and its use in reducing by one the number of degrees of freedom of the dynamical system.

The forty-five pages of Chapter IV on the solvable problems of particle dynamics form, when taken with the general theory of the preceding chapters, a veritable treatise on dynamics of a particle. From the simplest case of the pendulum the author goes through the various forms of central motion to motion on different kinds of surfaces. So many of the problems are soluble in finite form only in terms of the elliptic functions that some knowledge of these functions alike in the legendrian, the jacobian, and the weierstrassian analysis is indispensable. Fortunately the author has his own excellent work on modern analysis on which to fall back,* and references to it are frequent In this chapter there are about seventy problems, many of which should be interspersed with Chapters II and III, purely for pedagogic purposes. In fact, although the arrangement in the text is admirable for theoretical and logical purposes, instruction requires the interlarding of one part with another. There follows a short chapter of fifteen pages, with about as many problems, on the dynamical specification of bodies. It contains the usual theorems on moments, products, and ellipsoids of inertia, on kinetic energy, and on the independence of the motion of the center of gravity and of the motion relative to it. This leads directly to Chapter VI on the soluble problems of rigid dynamics — a complete analogue to Chapter IV as regards length, number of problems, and so forth. Naturally the top comes in for a lion's share of the space, and the elliptic functions are as indispensable as before.

Up to this point, about two fifths of the whole work has been covered with very little of a really advanced mathematical character except perhaps the technique of the elliptic functions. From here on, however, the amount of mathematics which the reader must know, or assimilate from the meager indications in the text, is by no means inconsiderable — calculus of variations, elementary divisors, integral invariants, and contact transforma-

* E. T. Whittaker, A Course in Modern Analysis, reviewed by Bôcher in the BULLETIN, vol. 10, pp. 351–354.

tions may be mentioned as examples. It is, then, just at this point that modern mechanics begins.

The seventh chapter of some thirty-five pages deals with the theory of vibrations whether about a state of equilibrium or of motion. The present method of treatment goes back to Weierstrass and requires a tolerable knowledge of the theory of quadratic forms, linear transformations, and elementary divisors. At times the analysis itself, apart from what it presupposes, is somewhat complicated. Nevertheless a good student even poorly equipped ought to get all the chief points with the aid of the forty or so problems which he may solve in connection with the text. The more difficult of these are a task for the best; but the easier are very simple and illustrate the principles about as well. It is remarkable how much mathematics that one does not fully understand will cheerfully be accepted on faith if only a number of problems come out successfully when the indicated method is applied—and this is often the best way to get to understand the mathematics.

Chapter VIII on non-holonomic systems and dissipative systems is admirable and admirably placed for theoretical purposes. In elementary and practical treatments of mechanics it is necessary to introduce such systems earlier because so many of the simplest systems are of this type; but the difficulties in the proper theoretical treatment of these systems render it advisable to postpone the discussion of them to a relatively later time. The clean-cut division of problems into holonomic and non-holonomic, which has become possible since the work of Hertz, is very desirable and has been adopted by more than one author. The neatness of treatment of dissipative systems is much enhanced by the timely introduction of Rayleigh's dissipation function. To be sure this function does not always exist; but in the simple and probably the most important case, that in which the resistances are proportional to the velocity, it does; and gives a noteworthy simplicity to the equations of motion. The collection of thirty-five or forty problems appears to be particularly well selected and certainly presents all grades of difficulty.

The principles of Hamilton and Gauss receive careful attention in the brief space of twenty pages. The reader will perhaps find some difficulty in appreciating the methods owing to the practical impossibility of finding more than a very few problems on which to try the text. This lack may be supplied

in part by having recourse to problems in earlier chapters. The author goes in some detail into the methods of the calculus of variations. He distinguishes between the applications of the principle of Hamilton to holonomic conservative systems, and its extension to dissipative and non-holonomic systems. It is in fact one of the admirable traits of this book to state with the maximum clearness just what are the conditions under which the different theorems and methods are obtained. Some treatises on dynamics merely give the equations in their simplest form and others, which presumably are more complete, fail to state the limitations under which the equations are derived. For reference purposes this is fatal. Whittaker's treatise is typographically and otherwise arranged so as to facilitate reference. To continue, however, with the present chapter we would state that the author takes up the questions of kinetic foci, of whether the integrals are actual minima (connected with the subject of fields and conjugate points in the parlance of the calculus of variations), and the way of representing dynamical systems by means of geodesics. The refinements of the modern rigorous treatment of the calculus of variations are, of course, out of the question in a work of this kind. A more natural use of space is the treatment of Bertrand's and Thomson's theorems with which the chapter closes.

The remaining one hundred and fifty pages of the book become successively more and more full of the most recent researches. Chapter X on hamiltonian systems and their integral invariants, Chapter XI on the transformation theory of dynamics with its important theorem that the whole course of a dynamical system can be regarded as the gradual self-unfolding of a contact transformation, and Chapter XII on the properties of the integrals of dynamical systems are all of tolerably recent development, although the germs of some of the methods are to be found in the works of Jacobi, Hamilton, Poisson, and even so far back as Lagrange. There are a few problems indicated for solution which must be warmly appreciated by all who know how hard it is to find specific problems in so new and advanced a domain. The last four chapters in the reduction of the problem of three bodies, the theorems of Bruns and Poincaré, the general theorem of orbits, and integration by trigonometric series are naturally essential parts of the field of mathematical astronomy, with which the most valuable part of the author's own researches have been connected. How much the

student of mechanics proper may be interested in this matter is a question. The advisability of including such a treatment here cannot, however, be well open to doubt. Mathematical astronomy as one conclusion of mechanics is quite in the spirit of the present time.

Personally we should have been happy to see introduced, before these pages crammed with necessarily complicated analysis, a treatment of two problems which are of prime importance in physics and without which no student of mechanics for its applications in physics rather than in astronomy can feel himself well equipped. The first is the study of the application of Lagrange's equations and the kinetic potential to problems in physics and chemistry. This was early introduced by Maxwell into electricity and magnetism, and is treated at length in J. J. Thomson's little book on the subject. It goes a long way toward giving the lagrangian function real physical interest. Such a chapter might properly find its place just before the tenth. The second problem is that of statistical mechanics which both in theory and its applications to the kinetic theory of gases and to thermodynamics is well available, though in too great detail for the débutant, in the works of Gibbs and Jeans. A chapter on this subject, placed perhaps right after the tenth, would go far on the way toward completing a treatise on dynamics in a direction likely soon to be of greater general interest than the special methods of astronomy. May we not hope that the author's interest in astronomy (he was recently appointed astronomer royal of Ireland, a position illustrious with great names) will not prevent him from giving adequate treatment of these two physical problems in later editions of his book?

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
April, 1906.

SOME RECENT FOREIGN TEXT BOOKS.

A Course in Practical Mathematics. By F. M. SAXELBY. London, Longmans, Green, and Co., 1905. 8vo. viii + 2 unnumb. + 430 pp. Price, $2.25.

Die Planimetrie für das Gymnasium. By GUSTAV HOLZMÜLLER. Leipzig and Berlin, B. G. Teubner, 1905. 8vo. viii + 240 pp. Price, M. 2.40.

Methodisches Lehrbuch der Elementar-Mathematik. By GUSTAV HOLZMÜLLER. Leipzig and Berlin, B. G. Teubner, 1903. 8vo. xiv+370 pp. Price, M. 4.40.

Vorbereitende Einführung in die Raumlehre. By GUSTAV HOLZMÜLLER. Leipzig and Berlin, B. G. Teubner, 1904. 8vo. x+123 pp. Price, M. 1.60.

IT is probably a fact that in America we look upon the educational system of England as rather unprogressive. It is also a fact that we are accustomed to look upon its mathematics as less interesting and fruitful than that of the other leading countries of Europe. In elementary work we think of the English school as so bound to Euclid, so given to algebras of the abstract, uninteresting Todhunter type, so repelled by the practical in its adherence to what it deems the cultural, that we feel we shall find nothing suggestive in this line on British soil. And so our teachers today hear more of what is done to improve mathematics and education in France, Germany, and Italy, than in the mother country to which we are so largely indebted for our early text books and curricula. At the present time we indulge in much talk and we write numerous essays upon making our elementary mathematics more practical, upon early setting forth such of the uses of the science as are within the range of the beginner's understanding, and upon relating mathematics to the other sciences and to the general life about us, and if we give the matter a passing thought we pity England because its educators have not awakened to this vital issue.

In view of this feeling on our part, it is interesting to see what England is really doing in the way of improving elementary education in general, and mathematics in particular. Ten years have seen a very marked change of view with respect to Euclid, and although the standard classical schools still use what seem to us very uninteresting algebras and geometries, it is to England that we have to go for some of the best elementary text books we have today on the applications of mathematics. We have talked a great deal, but English teachers have acted, and neither France nor Germany is facing the issue more earnestly.

Of the several elementary text books on applied mathematics that have appeared in the last few years, Mr. Saxelby's is by far the best. It is written for institutions of a class at present more highly developed in England than here, although our

technical schools are beginning to make themselves felt in a very serious way. It recognizes directly the utilitarian needs of these institutions, and in twenty-nine chapters, with less than four hundred pages, it presents the essential parts of plane trigonometry, graphs and analytic geometry, the differential and integral calculus, vector analysis, solid analytic geometry, and differential equations. This seems like a large domain, and the natural prejudgment would be that the work must be scrappy, unscientific, and incomplete. But after all, this depends upon the point of view. All elementary mathematics is scrappy, for the beginner cannot be expected to exhaust each subject before he proceeds to the next one; all is more or less unscientific, as when a pupil uses the binomial theorem before he knows anything about differentiation and convergency; and the treatment of no topic is complete. Thoroughness is a matter of degree.

Mr. Saxelby has condensed a great deal of theory into a small compass; he has eliminated what is not essential from the practical standpoint, and has arranged his material on a psychological rather than a logical plan. For example, he proves the addition formula for $\sin(A+B)$ in ten lines, but only for the case of A and B acute. The general case he postpones until he takes up vector analysis. This may not seem logical, but it is very psychological, and from the standpoint of teaching it is commendable. He does little with conics, which probably brings down upon his head much criticism from the traditional schoolmaster, but he investigates a great many curves such as the engineer and the statistician need. He does not study subnormals and subtangents, but he plots $i=50\sin 600t+20\sin 1800\,t$ for the purpose of studying an alternating electric current, and he considers an interesting set of curves representing compound periodic oscillations, and the educational results of such work are not to be discounted by the usual remark about utilitarianism.

The fact is that a student who fairly masters a book of this kind will like mathematics better than one who puts the same amount of time on our traditional courses; he will have a far better working knowledge of the applied science; he will have cultivated more originality; he will be more desirous of studying higher mathematics; and he will not have suffered materially in the pure field. Old school mathematicians may not like the idea of making applications the basis of selection, and they may be content with that most archaic of all our sets of

problems, the one generally found in our books on the calculus, but a reaction is bound to come in our college mathematics just as it has come in our school mathematics. The traditional boundaries between subjects must give way more and more, just as Mr. Saxelby has made them give way in his book; the founding of every step upon bed rock before the next step is taken must be recognized as the mission of a mature mathematician, not of a beginner who will probably not become a scientist; and the genuine applications of mathematics must replace the time-worn fictions of the 18th century writers.

This does not mean that Mr. Saxelby's work is a satisfactory text-book for American colleges. Some of his applications would not appeal to us; some of his symbolism is not ours; his text-book has not the appearance which our tastes demand; we have a different system of measures, in part, and we would probably be more traditional than he is in some respects, as in the study of certain quadratic curves. But books of this class are healthy ones for our college and preparatary teachers to have at hand; they breathe of the future; they stand for mathematics as a whole instead of its minor provinces, and for mathematics interrelated and extrarelated instead of isolated in all its parts. It is not improbable that the present agitation in the development of preparatory mathematics in this country will penetrate even into our colleges and stir up some new life. When this is done America will begin to develop for all of its schools what England is developing for its technical institutions, books of the spirit of Mr. Saxelby's, suited to the peculiar conditions and needs of our country.

In Germany of late no one has been more active in the revitalization of elementary mathematics than Professor Holzmüller. He does not represent the movement for which Mr. Saxelby stands, but his work is none the less suggestive and valuable. Germany has never been a slave to Euclid, nor have even the most conservative teachers in that country ever agreed to the interpretation of the dogma of thoroughness which has made much of English and American mathematics so repulsive. It is well known that the German schools begin their algebra and geometry much earlier than we, and extend them over a longer period, adopting a parallel instead of a tandem arrangement. It is in this elementary field, and with these principles to guide him, that Professor Holzmüller has worked.

His Planimetrie is designed for the Quarta to the Untersekunda, say the sixth to the ninth school years. As usual with German elementary text-books, it pays relatively little attention to formal definitions, but devotes much care to developing clear concepts; it is not particularly concerned with what propositions are discussed, but rather as to how they are treated; and it is not so much moved by the desire for a logical sequence as by the hope that interest, originality, and correct ideas may result from the study. As Professor Holzmüller says in his Einführung in die Raumlehre, "Erst methodische Auswahl, dann System! Erst Pestalozzi, dann Euklid!"

Brought up as we are on some revised version of Euclid, and accustomed to the traditional nomenclature, it seems strange to us to see a statement like this for a sixth-grade child: "Eine gerade Punktreihe ist eine solche, die dem Auge durch einen einzigen Punkt verdeckt werden kann." We would be apt to say either that it was too advanced for a child of eleven or twelve, or that it was not sufficiently scientific, depending upon our point of view. And yet this is a type of the statements developed for these beginners. The work is at first largely constructive, with proofs as the capabilities of the child permit. Following the advanced tendencies set forth by Méray and De Paolis there is no distinction drawn between plane and solid geometry. In the sixth school year the tetrahedron is briefly studied before the concept of angle is introduced, and most of the essential propositions of Euclid I are proved along with some treatment of elementary solids. In the Unter- and Obertertia, the seventh and eighth school years if we allow three years before the Gymnasium, plane geometry is fairly well covered, with the exception of the doctrine of similarity. Some of the work laid down would be considered too difficult for our average high schools, particularly the propositions involving cross quadrilaterals and the development of Heron's formula for the triangle. The work of Untersekunda, say the first year in our high school in point of time, relates to similar figures, the radical axis, inversion, elementary projections for map drawing, the regular solids, and the evaluation of π.

The Methodisches Lehrbuch der Elementar-Mathematik, Dritter Teil, now in its second edition, is intended for the Real and Fachschulen, and hence it involves a line of work not found in the Planimetrie. It is an excellent preparation for the study of modern mathematics, combining as it does the essen-

tials of the recent geometry of the triangle, the theory of duality, a little projective geometry, propositions like Feuerbach's and Malfatti's on the circle and Pascal's on the hexagram in a conic, the theory of geometric involution, the theory of conics with its application to physics, stereometry, axonometry, spherics, and the application of higher algebra to geometry. Compared with Mr. Saxelby's work it involves much more mathematics and it is less practical; but on the other hand it is practical in certain directions which are not found in the English work, particularly in the theory of electricity, in higher mensuration, and in cartography. The book is one to be recommended to teachers who wish a brief general introduction to modern mathematics.

Professor Holzmüller's Vorbereitende Einführing in die Raumlehre has not so much to commend it as his other works just mentioned. It is a good book for teachers who wish to supplement the regular work in geometry by interesting side lights on the subject. Such topics as the pythagorean triangle numbers, the geometric proof concerning the sum of n odd numbers, the solids inscriptible in the regular solids, and the various types of semi-regular and stellar polyhedra are here discussed with a simplicity that renders them usable in the school room.

It should be said that none of Professor Holzmüller's works are text-books in the American sense. This is, of course, understood by all who have visited German schools or have read Professor Young's excellent work on the subject. Our type of text-book is practically unknown in the country where the pupils study and recite in the same class period and always under the master's guidance. They are, however, none the less suggestive to the American teacher, for they show what can be accomplished with pupils in certain grades, what can be omitted in the preparation of students from whom some of the world's best mathematicians are being made, what modern mathematics can replace those portions of the ancient science that are eliminated, what attention can be given to applications without interfering with the vague idea of culture which so often possesses us, and what correlation is possible among the various traditionally separate fields of the science of mathematics.

DAVID EUGENE SMITH.

NOTES.

AT the meeting of the London mathematical society held on April 26 the following papers were read: By E. B. ELLIOTT, "Perpetuants and contraperpetuants"; by A. C. DIXON, "A question in the theory of aggregates," and "The canonical forms of the ternary sextic and quaternary quartic;" by P. E. B. JOURDAIN, "On the question of the existence of transfinite numbers"; by G. H. HARDY, "Some theorems connected with Abel's theorem on the continuity of power series"; by A. R. RICHARDSON, "On a set of intervals about the rational numbers."

THE American philosophical society celebrated April 17–20 the two hundredth anniversary of the birth of its founder, Benjamin Franklin. Nearly one hundred and fifty learned societies and universities were represented at the celebration, Professor E. V. HUNTINGTON being the delegate of the AMERICAN MATHEMATICAL SOCIETY. Among the scientific papers which were presented, the following were of special interest to mathematicians: "The figure and stability of a liquid satellite," by G. H. DARWIN; "The present position of the problem concerning the first principles of scientific theory," by JOSIAH ROYCE; "Form analysis," by A. A. MICHELSON; "On positive and negative electrons," by H. A. LORENTZ.

THE next annual meeting of the Deutsche Mathematiker-Vereinigung will be held in affiliation with the seventy-eighth convention of the society of German naturalists and physicians at Stuttgart, September 16–22, under the presidency of Professor A. PRINGSHEIM, of Munich. While papers on other subjects are acceptable, principal emphasis will be put upon the theory of functions; various reports have been provided for, particularly of functions of several variables. Titles and abstracts of papers should be immediately sent to the secretary, Professor A. KRAZER, Karlsruhe in Baden, Westendstrasse 57.

THE sixth meeting of the Association of teachers of mathematics in the Middle States and Maryland was held at Columbia University on April 14. The New York section met at the same place on May 12. The Association will undertake the preparation of a bibliography of mathematical works useful to teachers. A similar undertaking was carried into effect some time ago by the Central Association. The Missouri Society of

teachers of mathematics, at a meeting held at Columbia, Mo., on May 5, decided to enlarge its scope to include the sciences. Mr. J. W. WITHERS was elected president for the coming year. Provision was made for representation in the organization of a national society and for cooperation with similar societies in the discussion of the teaching of elementary physics.

THE Colorado mathematical society was organized last November, Professor I. M. DeLong being the first president. Meetings are held at Denver at intervals of about six weeks.

THE publishing house of John Wiley and Sons, New York, announce the following mathematical books in press: "Introduction to infinitesimal analysis and functions of one real variable," by O. VEBLEN and N. J. LENNES; "Textbook of mechanics," by L. A. MARTIN, volume one, Statics, volume two, Kinematics and kinetics; "Irrational numbers and their representation by sequences and series," by H. P. MANNING; "Mathematical monograms," edited by M. MERRIMAN and R. S. WOODWARD.

AMONG the books in press of the publishing house of B. G. Teubner in Leipzig, may be mentioned: Encyclopädie der mathematischen Wissenschaften, Band III, 1, Prinzipien der Geometrie, by F. ENRIQUES, Die Begriffe Linie und Fläche, by H. v. MANGOLDT, Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwickelung, by G. FANO; III, 2, Abzählende Methoden, by H. G. ZEUTHEN, Allgemeine theorie der höheren ebenen algebraischen Kurven, by L. BARZOLARI; IV, 2, Besondere Ausführungen über unstetige Bewegungen in Flüssigkeiten, by G. ZEMPLÉN, Hydraulik, by PH. FORCHHEIMER, Theorie der hydraulischen Motore und Pumpen, by M. GRÜBLER, Allgemeine Theorie der Elastizitatät, by O. TEDONE, and five articles by joint authors for volume five, parts one and two. The first three volumes of tome one of the French edition are also in press.

THE Randolph Jones company of Chicago have prepared for sale a large number of Hanstein's model and goniostat for aid in the teaching of plane and solid geometry, perspective, shadow construction, etc.

AT the University of Berlin a prize of 1000 marks has been offered for the best doctor thesis in mathematics and physics submitted to the University in 1906.

THE philosophical faculty of the University of Halle announces the following prize problem:

It is known that the determination of surfaces having constant mean curvature depends upon the same differential equation as the determination of surfaces of constant negative curvature. A number of the latter surfaces are known, and also a number of solutions of the associated partial differential equation. The surfaces of constant mean curvature which belong to these solutions are to be investigated.

Competing manuscripts should be submitted to the Secretary of the University, under the usual conditions, before October 27, 1906. The prize of 150 Marks will be awarded January 27, 1907.

THE second section of the philosophical faculty of the University of Zürich announces the following prize problem:

The motion of a material point on the surface of a sphere is to be investigated by means of at least two of the more modern methods of astronomical mechanics and the results, both analytic and numerical, compared with those obtained by means of elliptic functions.

Competing memoirs should be sent to the Rector of the University under the usual conditions by December 31, 1906.

THE following advanced courses in mathematics are announced to be given at the summer session, July 5–August 16, of the respective universities:

COLUMBIA UNIVERSITY: By Professor JAMES MACLAY: Elliptic functions. — By Professor C. J. KEYSER; Modern analytical theories in geometry. — By Professor M. W. HASKELL: Integral calculus. — By Professor EDWARD KASNER: Differential calculus.

UNIVERSITY OF PENNSYLVANIA. — By Professor G. E. FISHER: Invariants and covariants. — By Professor J. I. SCHWATT: Definite integrals. — By Professor G. H. HALLETT: Theory of abstract groups. — By Dr. F. H. SAFFORD: Differential equations.

The following advanced courses in mathematics are announced for the academic year 1906–1907:

COLUMBIA UNIVERSITY. — By Professor T. S. FISKE: Advanced calculus, introduction to the theory of functions of a

real variable, three hours; Functions defined by linear differential equations, three hours. — By Professor F. N. COLE: Introduction to the theory of functions, three hours; Theory of plane curves, three hours. — By Professor D. E. SMITH: History of mathematics, two hours. — By Professor JAMES MACLAY: Application of the calculus to the theory of surfaces, three hours; Elliptic functions, three hours. — By Professor C. J. KEYSER: Modern theories in geometry, three hours; Principles of mathematics, three hours. — By Professor H. B. MITCHELL: Elements of quaternions, two hours; Differential equations, two hours. — By Professor EDWARD KASNER: Partial differential equations, two hours. — By Dr. G. H. LING: Theory of numbers, three hours; Advanced theory of numbers, three hours. — By Professor M. I. PUPIN: Theory of the potential function, two hours; Partial differential equations of physics, two hours; Hydrokinetics, two hours; Special problems, two hours. — By Professor A. P. WILLS: Mechanics, two hours; Theory of elasticity, two hours; Electricity and magnetism, electromagnetic theory of light, two hours; Thermodynamics, two hours.

Special courses in mathematical physics, of about twelve lectures each, will be given by Professor J. LARMOR, of the University of Cambridge, and Professor O. LUMMER, of the University of Breslau. Dates of these lectures and further details will be announced later.

CORNELL UNIVERSITY. — By Professor L. A. WAIT: Advanced analytic geometry, three hours; Differential calculus, II, two hours. — By Professor G. W. JONES: Algebra and imaginaries, three hours. — By Professor J. MCMAHON: Mechanics and hydrodynamics, two hours; Fourier's series and spherical harmonics, three hours. — By Professor J. H. TANNER: Theory of equations, two hours. — By Professor J. I. HUTCHINSON: Projective geometry, three hours; Seminar in automorphic functions, two hours. — By Professor V. SNYDER; Algebraic plane curves, three hours; Definite integrals, two hours. — By Professor W. B. FITE: Theory of functions of a complex variable, three hours; Theory of groups, two hours (first half year); Theory of assemblages, two hours (second half year).

HARVARD UNIVERSITY. — By Professor W. E. BYERLY:

Dynamics of a rigid body, three hours. — By Professors W. E. BYERLY and B. O. PEIRCE: Trigonometric series, introduction to spherical harmonics, the potential function, three hours. — By Professor B. O. PEIRCE: Hydromechanics, two hours. — By Professor W. F. OSGOOD: Differential and integral calculus, second course, three hours; Infinite series and products, three hours, first half-year; Algebra, Galois's theory of equations, three hours, second half-year. — By Professor M. BÔCHER: Methods of interpolation and approximation, three hours, first half-year; Theory of numbers, three hours, second half-year; Linear differential equations of physics, three hours. — By Professor C. L. BOUTON: Differential equations, with an introduction to Lie's theory of continuous groups, three hours. — By Mr. J. K. WHITTEMORE: Theory of functions (introductory course), three hours; Introduction to the differential geometry of curves and surfaces, three hours, second half-year. — By Dr. J. L. COOLIDGE: Introduction to modern geometry and modern algebra, three hours; Line geometry, three hours. — By Professor E. V. HUNTINGTON: Short course on the fundamental concepts of mathematics. — Courses in reading and research are offered by Professor BYERLY, on Some famous problems in mechanics; by Professor OSGOOD, on Topics in the theory of functions; by Professor BÔCHER, on Fluctuating functions; by Professor BOUTON, on Topics in the theory of contact transformations; by Mr. WHITTEMORE, on Goursat's Cours d'analyse; by Dr. COOLIDGE, on Enriques's Projective geometry. — The mathematical conference will meet as usual.

UNIVERSITY OF BONN. — By Professor E. STUDY: Analytic geometry; II (projective geometry), three hours; Introduction to the theory of invariants, three hours; Seminar, two hours. — By Professor G. KOWALEWSKI: Introduction to the theory of numbers, two hours; Differential and integral calculus, four hours; Critical survey of new results in the theory of assemblages, one hour; Seminar, two hours. — By Professor F. LONDON: Descriptive geometry, four hours; Theory of elliptic functions, four hours; Seminar, two hours.

THE Special Board for Mathematics have issued a report with proposals which, if carried, will have the effect of altering the character of that time-honored institution, the mathematical tripos in Cambridge (England). At present the better stu-

dents take the examinations at the end of their third and fourth years, that for the latter consisting of advanced subjects only. It is proposed to divide the examinations into two parts, one of which would be taken by these men at the end of their first year: it would consist of elementary mathematics, including the calculus and the elements of dynamics, electricity and optics. The second part would be taken at the end of the third year: it is divided into two schedules. Schedule A consists of more advanced work in the same subjects and in addition, elementary theory of functions, differential equations, hydromechanics and astronomy; Schedule B contains the higher subjects. The men would therefore complete their course at the end of the third year instead of at the end of the fourth year; the additional time could be given to study without the compulsion of an examination. Students in physics and engineering would naturally take part I and schedule A of part II.

A feature of the proposals is to abolish the order of merit, arranging the names in three classes only, and attaching a special mark to the names of those who satisfy the examiners in schedule B and another special mark to those who deserve special credit in the subjects under this schedule.

Some quotations from the report may be of interest. The Board considers that under the old scheme "the ablest men . . . are encouraged to spend an excessive amount of time upon detailed work in the less advanced parts of mathematics." Again they spend "much time during the first three years of their course upon an excessive amount of polishing of their mathematical tools"; "leave themselves an insufficient amount of time for reaping the advantages of the skill they have acquired, by applying that skill to more advanced work." The educational side for others than professed mathematicians is also carefully considered.

At the technical school of Berlin, Dr. M. Reissner has been appointed associate professor of mechanics and Dr. G. Wallenberg has been appointed docent in pure mathematics.

Dr. C. v. Wisselingh has been appointed professor of mathematics at the University of Groningen.

Professor J. A. Gmeiner, of the German University at Prague, has been appointed professor of mathematics at the University of Innsbruck, as successor to the late Professor O. Stolz.

PROFESSOR E. B. VAN VLECK, of Wesleyan University, has been appointed head professor of mathematics at the University of Wisconsin to succeed Professor C. A. VAN VELZER.

PROFESSOR J. A. MILLER, of the University of Indiana, has been appointed professor of mathematics and astronomy at Swarthmore College.

PROFESSOR G. A. MILLER, of Stanford University, has been appointed associate professor of mathematics at the University of Illinois.

PROFESSOR S. E. SLOCUM, of the University of Illinois, has been recalled to the University of Cincinnati, as professor of applied mathematics in the college of engineering.

PROFESSOR G. A. BLISS, of Princeton University, will become a member of the editorial board of the *Annals of Mathematics* with the beginning of the next volume in October, succeeding Professor H. S. WHITE, who retired last fall.

DR. C. N. HASKINS, of Cornell University, has been appointed assistant professor of mathematics at the University of Illinois.

AT the Massachusetts Institute of Technology the following changes are announced: Professor W. WELLS and Mr. W. S. MACDONALD have been granted leave of absence for next year; Professor F. S. WOODS has been promoted to a full professorship and Mr. N. R. GEORGE to an assistant professorship of mathematics.

PROFESSOR W. B. SMITH, of Tulane University, has been transferred from the chair of mathematics to a chair in philosophy.

DR. J. G. HARDY has been promoted to an associate professorship of mathematics at Williams College.

AT the University of Colorado Dr. SAUL EPSTEEN has been promoted to an assistant professorship of mathematics.

DR. C. E. COLPITTS, of Cornell University, has been appointed adjunct professor of mathematics at the Georgia School of Technology.

MR. L. P. SICELOF has been appointed tutor in mathematics at Columbia University.

PROFESSOR J. H. TANNER, of Cornell University, who has been absent on leave during the present academic year, will resume his work at the university in September.

MR. C. F. CRAIG has been appointed assistant in mathematics at Cornell University.

DR. O. E. GLENN has been appointed instructor in mathematics at the University of Pennsylvania.

RECENT catalogues of second-hand mathematical books: Theodor Ackermann, 10 Promenadenplatz, Munich, catalogue No. 550, 662 titles in mathematics and physics. Williams and Norgate, 14 Henrietta Street, Covent Garden, London, general science including journals, about 90 titles in mathematics. Mayer and Müller, Prinz Louis Ferdinandstrasse, 2, Berlin, N. W., catalogue No. 220, 142 pages on mathematics, physics, and astronomy; W. Heffer and Sons, Cambridge, England, catalogue No. 16, 2722 titles on mathematics, physics, and astronomy; Jacques Rosenthal, Karlstrasse, 10, Munich, catalogue No. 42, Bibliotheca Paedogogica, Part I, containing many early mathematical works. From May 21 to May 24, 1906, Southeby, Wilkinson and Hodge, of London, sold at auction the library of R. C. Fisher, Esq., which contained many Rara Mathematica, including the Treviso Arithmetic of 1478.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALAS (J. M. DE LAS). Determinantes. Toledo, Gómez Menor, 1905. 29 pp. P. 3.00

BAFFI (C.). Sulle generazioni dei complessi tetraedrali. Bologna, Cuppini, 1905. 8vo. 9 pp.

BIASI (G.). Intorno alle trasformazioni cremoniane e ad una geometria analitica di grado superiore che ne deriva. Sassari, Gallizzi, 1905. 8vo. 27 pp.

BÔCHER (M.). On harmonic functions in two dimensions. 8vo. (*Proceedings of the American Academy of Arts and Sciences*, Vol. XI, pp. 577–583.)

BOWSER (E. A.). Elementary treatise on the differential and integral calculus, with numerous examples. 21st edition, enlarged by 640 additional examples. New York, Van Nostrand, 1906. 12mo. 460 pp. Cloth. $2.25

CARRUS (S.). Familles de surfaces à trajectoires orthogonales planes. (Thèse.) Paris, Gauthier-Villars, 1906. 4to. 93 pp.

CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. Band I, 2te Hälfte. 2te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1906. 8vo. 14 + 257–560 pp. M. 6.00

HARTWIG (T.). Schule der Mathematik zum Selbstunterrichte. (Neue mathematische Unterrichtsbriefe.) 3ter Band: Differential- und Integralrechnung. Wien, Perles, 1906. 8vo. 200 pp. M. 2.50

HILL (G. W.). Collected mathematical works. (In 4 volumes.) Vol. III. Washington, Carnegie Institution, 1906. 4to. 577 pp. Cloth. $2.50

NEIKIRK (L. I.). Groups of order p^m which contain cyclic subgroups of order p^{m-3}. (*Publications of the University of Pennsylvania*, Series in mathematics.) 8vo. 65 pp. $0.75

PIRONDINI (G.). Contributo alla teoria delle serie: Memoria. Parma, Rossi-Ubaldi, 1905. 8vo. 32 pp.

POMPEIU (D.). Sur la continuité des fonctions de variables complexes. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 53 pp. Fr. 3.00

RILLY (A.). Le problème du cavalier des échecs. Etude sur la symétrie latérale en deux chaînes fermées. Paris, Rilly, 1906. 8vo. 108 pp.

SACHS (J.). Projektivische (neuere) Geometrie. Nach System Kleyer bearbeitet. (3 Teile.) Teil III: Pol und Polare; Mittelpunktseigenschaften; Involution; Brennpunktseigenschaften der Kurven 2ten Grades. (Etwa 25 Hefte.) Heft I. Bremerhaven, 1906. 8vo. M. 0.25

SIMON (M.). Analytische Geometrie der Ebene. (Sammlung Göschen, 65.) 2te, verbesserte Auflage. Leipzig, Göschen, 1906. 8vo. 196 pp. Cloth. M. 0.80

TANNERY (J.). Leçons d'algèbre et d'analyse, à l'usage des élèves des classes de mathématiques spéciales. Tome 2. Paris, Gauthier-Villars, 1906. 8vo. 640 pp. Fr. 12.00

ZORETTI (L.). Sur les fonctions analytiques uniformes qui possèdent un ensemble parfait discontinu de points singuliers. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 53 pp.

II. ELEMENTARY MATHEMATICS.

ARZELÀ (C.). Trattato di algebra elementare ad uso dei licei e degli istituti tecnici. 3a edizione, completamente rifatta. Firenze, 1906. 16mo. 15 + 503 pp. L. 3.50

BOURLET (C.) ET HULOT (J.). Précis d'algèbre, rédigé conformément aux programmes du 27 juillet 1905. Paris, Hachette, 1906. 16mo. 507 pp. Fr. 3.50

BUCK (R. C.). A manual of trigonometry. 3rd edition, revised and corrected. London, Griffin, 1906. 8vo. 124 pp. 3s. 6d.

CRACKNELL (A. G.). See WORKMAN (W. P.).

DICKNETHER (F.). Lehrbuch der Arithmetik nebst Uebungsaufgaben für Mittelschulen. 2ter Teil. 2te Auflage. München, Lindauer, 1906. 8vo. 4 + 125 pp. M. 1.80

FAIFOFER (A.). Elementi di algebra ad uso della prima classe liceale. Venezia, Sorteni, 1905. 16mo. 193 pp. L. 2.00

GAZZANIGA (P.). See VERONESE (G.).

HAAS (A.). Der binomische und polynomische Lehrsatz, die arithmetischen Reihen höherer Ordnung und die unendlichen Reihen. Nach System Kleyer bearbeitet. Teil I: Der binomische Lehrsatz für ganze positive Exponenten. (Etwa 27 Hefte.) Heft 1. Bremerhaven, 1905. 8vo. M. 0.25

HOLTZE (A.). See KÖSTLER (H.).

HULOT (J.). See BOURLET (C.).

KOHLSCHÜTTER (E.). Messkarte zur Auflösung sphärischer Dreiecke. Nach Chauvenet neu herausgegeben. Berlin, 1906. M. 1.50

KÖSTLER (H.). Leitfaden der ebenen Geometrie für höhere Lehranstalten, neu herausgegeben von A. Holtze. 1tes Heft: Kongruenz. 6te, teilweise umgearbeitete Auflage. Halle, Nebert, 1906. 8vo. 86 pp. M. 1.35

LALANDE (J. DE). Tables des logarithmes. Avec additions et modifications par E. E. Neël. 10e édition. Louvain, 1905. 8vo. 210 pp. Fr. 3.50

LEGENDRE (A. M.). Compendio di geometria: libro di testo per le scuole tecniche conforme ai programmi governativi, ricavato dagli Elementi di A. M. Legendre per cura di G. Tolomei. Firenze, Ricci, 1905. 8vo. 179 pp. L. 1.75

LEMAIRE (G.). Méthodes de résolution et de discussion des problèmes de géométrie. 2e édition. Paris, Vuibert, 1906. 8vo. 227 pp.

LOMBARDO (S. G.). Elementi di geometria per gli aspiranti alla reale scuola militare di Modena. Messina, Davì, 1906. 16mo. 101 pp. L. 1.50

LÜBSEN (H. B.). Ausführliches Lehrbuch der Arithmetik und Algebra zum Selbstunterricht und mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. 26te Auflage. Leipzig, Brandstetter, 1906. 8vo. 6 + 261 pp. M. 4.50

NEËL (E. E.). See LALANDE (J. DE).

PURSER (F.). Elementary geometry based on Euclid's Elements. London, Longmans, 1906. 12mo. 2s. 6d.

SASSO (M.) Formole della quarta e quinta potenza dei polinomi e loro applicazione. Avellino, Pergola, 1905. 8vo. 21 pp.

SIMON (M.). Methodik der elementaren Arithmetik in Verbindung mit algebraischer Analysis. Leipzig, Teubner, 1906. 8vo. 108 pp. M. 3.20

SIMÓN Y MAYORGA (J.). Nociones de geometría, seguidas de una colección de ejercicios y problemas. Salamanca, Iglesias, 1906. 114 pp. P. 2.50

SOCCI (A.) E TOLOMEI (G.). Nozioni intuitive di geometria elementare: libro di testo per il ginnasio inferiore secondo i vigenti programmi. 2a impressione. Firenze, Fiorentina, 1906. 8vo. 112 pp. L. 1.00

TOLOMEI (G.). See LEGENDRE (A. M.), and SOCCI (A.).

VAGNIER et VERNADET. Algèbre. Paris, Delagrave, 1906. 8vo. 275 pp.

VERNADET. See VAGNIER.

VERONESE (G.). Elementi di geometria ad uso dei ginnasi e licei e istituti tecnici, trattati con la collaborazione di P. Gazzaniga. Parte II. 3a edizione. Padova, Prosperini, 1905. 8vo. 220 pp. L. 2.25

WELLS (W.). Algebra for secondary schools. Boston, Heath, 1906. 12mo. 10 + 462 pp. Half leather. $1.20

WORKMAN (W. P.) and CRACKNELL (A. G.). Preparatory course in geometry. 2nd edition. London, Clive, 1906. 8vo. 76 pp. 9d.

III. APPLIED MATHEMATICS.

AALST (J. W. VAN). Theoretische beschouwingen over het warmteproces in stoomturbines, petroleum- en gasmotoren en verkoelingsmachines, met korte beschrijving van enkele soorten dier machines. Helder, 1905. 8vo. 4 + 119 pp. M. 3.00

ANCEY (C.). Théorie et pratique des opérations d'assurance. 2e édition, entièrement refondue et complétée. Paris, 1906. 8vo. 437 pp. Fr. 7.00

ANTOMARI (X.). Cours de géométrie descriptive. 3e édition, conforme au nouveau programme de mathématiques spéciales. Paris, Vuibert, 1906. 8vo. 623 pp.

BISCAN (W.). Die Starkstromtechnik. (2 Bände.) Band I: Gesetze und Erzeugung der elektrischen Energie. Leipzig, 1906. 8vo. 10 + 488 pp. M. 15.00

BROGGI (U.). Matematica attuariale. Storia statistica della mortalità. Matematica delle assicurazioni sulla vita. Milano, 1906. 8vo. 15 + 347 pp. L. 3.50

CAPPA (S.). Corso di meccanica applicata alle macchine. Torino, 1905. 8vo. 658 pp. L. 32.50

CLOUTH (F. M.). Tables pour le calcul des coordonnés goniométriques. 3e édition, revue et corrigée. Halle, Nebert, 1906. 8vo. 8 + 201 pp. M. 7.00

——. Tafeln zur Berechnung goniometrischer Coordinaten. 3te, neu bearbeitete Auflage. Halle, Nebert, 1906. 8vo. 8 + 201 pp. Cloth. M. 7.50

COLLET (J.) Compensation des figures géodésiques. Théorie et applications. Paris, Gauthier-Villars, 1905. 8vo. 44 pp. Fr. 1.50

DARIÈS (G.). Calcul des conduites d'eau. 2e édition. Paris, Gauthier-Villars, 1905. 8vo. 192 pp. Fr. 3.00

DONATI (L.). Lezioni di fisica matematica, raccolte a cura di R. Bonola e R. Viti. Bologna, 1905. 8vo. 352 pp. L. 10.00

FÖRSTER (R. VON). Ueber die Asymmetrie der Ablenkungen und ihren Zusammenhang mit der Asymmetrie der Schwingungen bei einem magnetischen Horizontalintensitätsvariometer. (Diss.) Marburg, 1905. 8vo. 35 pp.

FROHN (C.). Die graphische Statik. Zum Gebrauche an technischen Unterrichtsanstalten, zum Selbststudium und für die Bureaupraxis bearbeitet. Leipzig, Voigt, 1906. 8vo. 6 + 96 pp. M. 4.50

GOUILLY (A.). Traité de mécanique élémentaire. Bâle, 1906. 8vo. 220 pp. Fr. 5.50.

GUILHAUMON (J. B.). Eléments de cosmographie et de navigation. Précédés de notions de trigonométrie sphérique et d'optique. 4e édition. Nancy, 1906. 8vo. 489 pp. Fr. 8.00

HARTWICH (A.). Die magnetische Energie. (Progr.) Antwerpen, 1905. 4to. 43 pp.

HAYFORD (J. F.). See WRIGHT (T. W.).

HUPFELD (H.). Studien über Strömungsströme. (Diss.) Göttingen, 1905. 8vo. 47 pp.

INTERMEDIATE science papers of applied mathematics. Questions set at the University of London from 1867 to 1905. London, 1906. 8vo. 68 pp. 2s. 6d.

JÄGER (G.). Die Fortschritte der kinetischen Gastheorie. (Sammlung naturwissenschaftlicher und mathematischer Monographien, 12tes Heft.) Braunschweig, Vieweg, 1906. 8vo. 11 + 121 pp. M. 4.10

JAMES (E.). Théorie et pratique de l'horlogerie à l'usage des horlogers et des écoles d'horlogerie. Paris, Gauthier-Villars, 1906. 16mo. 6 + 228 pp. Fr. 5.00

KLAUS (A.). Ueber die Entwicklung der kinetischen Gastheorie und ihre Bedeutung für die moderne Physik. Freiburg, 1904. 8vo. 91 pp. M. 2.00

KLIMPERT (R.). Akustik oder die Lehre vom Schall. Nach System Kleyer bearbeitet. (3 Bände.) Band II: Die verschiedenen Tonerreger. (In circa 32 Heften.) Heft 1. Bremerhaven, 1906. 8vo. M. 0.25

LORENTZ (H. A.). Lehrbuch der Physik. Zum Gebrauche bei akademischen Vorlesungen. Nach der 4ten, von H. A. Lorentz und L. H. Siertsema bearbeiteten Auflage und unter Mitwirkung des Verfassers aus dem Holländischen übersetzt von G. Siebert. 1er Band. Leipzig, Barth, 1906. 8vo. 5 + 482 pp. Cloth. M. 9.00

——. Beginselen der Naturkunde. 4te druk, bewerkt door H. A. Lorentz en L. Siertsema. (2 Teile.) Leiden, 1905. 8vo. M. 17.00

——. Ergebnisse und Probleme der Elektronentheorie. 2te Auflage. Berlin, 1906. 8vo. 59 pp. M. 1.50

MARX (E.). See THOMSON (J. J.).

MELLOR (J. W.). Höhere Mathematik für Studierende der Chemie und Physik und verwandter Wissensgebiete. In freier Bearbeitung der 2ten englischen Ausgabe herausgegeben von A. Wogrinz und A. Szarvassi. Berlin, Springer, 1906. 8vo. 11 + 412 pp. M. 8.00.

MIE (J.). Die neueren Forschungen über Ionen und Elektronen. 2te Auflage. Stuttgart, 1906. 8vo. 40 pp. M. 1.20

RIPPER (W.). Steam engine theory and practice. 4th edition, revised and enlarged. New York, Longmans, 1905. 8vo. 12 + 452 pp. $2.50

RITTER (W.). Anwendungen der graphischen Statik. Nach C. Culmann bearbeitet. 4ter Teil: Der Bogen. Zürich, Raustein, 1906. 8vo. 7 + 269 pp. Cloth. M. 10.60

RÒITI (L.). Nozioni di resistenza dei materiali. Torino, Streglio, 1905, 16mo. 239 pp.

SIEBERT (G.). See LORENTZ (H. A.).

SIERTSEMA (L. H.). See LORENTZ (H. A.).

SIEVEKING (H.). Beiträge zur Theorie der elektrischen Entladung in Gasen. Karlsruhe, 1906. 8vo. 71 pp.

SOMMERFELDT (E.). Geometrische Kristallographie. Leipzig, Engelmann. 1906. 8vo. 10 + 139 pp. M. 7.00

STIATTESI (R.). Nuove formule per la determinazione della distanza degli epicentri sismici coi dati dei sismogrammi. (Osservatorio geodinamico di Quarto.) Firenze, Ricci, 1905. 8vo. 8 pp.

SZARVASSI (A.). See MELLOR (J. W.).

TAPLA (T.). Grundzüge der niederen Geodäsie. (3 Teile.) Teil III: Kartierung. Wien, 1906. 8vo. 7 + 107 pp. M. 3.50

THOMSON (J. J.). Elektrizitäts-Durchgang in Gasen. Deutsche autorisierte Ausgabe unter Mitwirkung des Autors besorgt und ergänzt von E. Marx. 2te und 3te Lieferung. Leipzig, Teubner, 1906. 8vo. Pp. 217–587. Cloth. M. 19.00

WOGRINZ (A.). See MELLOR (J. W.).

WRIGHT (T. W.). The adjustment of observations by the method of least squares, with applications to geodetic work. With the coöperation of J. F. Hayford. 2d edition. New York, Van Nostrand, 1906. 8vo. 9 + 298 pp. Cloth. $3.50